Complementi 6 - Richiami di analisi vettoriale

[Ultima revisione: 16 gennaio 2009]

Sia B un insieme aperto di R^3. Una funzione ϕ che assegna ad ogni punto x ∈ B uno scalare ϕ(x) si dira' un *campo scalare* su B. Analogamente, una funzione ϕ che assegna ad ogni punto x ∈ B un vettore ϕ(x) oppure un tensore ϕ(x) si dira' un *campo vettoriale*, o un *campo tensoriale* su B, rispettivamente. In questo notebook si forniscono le defnizioni di gradiente, divergenza, rotore e laplaciano per un campo vettoriale e per un campo tensoriale. Si introduce poi l'operatore di incompatibilita', che permette di esprimere le equazioni di compatibilita' di de Saint-Venant in forma compatta.

Alcune definizioni

I campi vettoriali

Sia **u** un *campo vettoriale* su B, e si supponga che **u** sia differenziabile nel punto x ∈ B. Si definiscono le seguenti quantita':

a) il *gradiente* di **u** nel punto x e' pari a:

$$(\nabla u(x))_{ij} = \frac{\partial u_i}{\partial x_j}(x) = u_{i,j}(x) \qquad (1)$$

o, esplicitamente:

$$\nabla u = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix} \qquad (2)$$

b) la *divergenza* di u nel punto x e' uno scalare, fornito da:

$$\operatorname{div} u(x) = \operatorname{tr} \nabla u(x) \qquad (3)$$

ossia, indicialmente:

$$\operatorname{div} u = u_{i,i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \qquad (4)$$

c) il *rotore* di un campo vettoriale **u** nel punto x e' un ulteriore vettore, denotato curl **u**, e' definito come il doppio del vettore assiale corrispondente alla parte antisimmetrica di ∇u(x), ossia e' l'unico vettore per cui:

$$(\nabla \mathbf{u}(x) - \nabla \mathbf{u}(x)^T)\,\mathbf{a} = (\operatorname{curl} \mathbf{u}(x)) \times \mathbf{a} \qquad (5)$$

per ogni vettore **a**.

Indicialmente si ha:

$$[\mathrm{curl}\, \mathbf{u}\, (x)]_i = \epsilon_{ijk}\, u_{k,j}\, (x) \tag{6}$$

o, per esteso:

$$[\mathrm{curl}\, \mathbf{u}\, (x)]_1 = u_{3,2}\, (x) - u_{2,3}\, (x) \tag{7}$$

$$[\mathrm{curl}\, \mathbf{u}\, (x)]_2 = u_{1,3}\, (x) - u_{3,1}\, (x) \tag{8}$$

$$[\mathrm{curl}\, \mathbf{u}\, (x)]_3 = u_{2,1}\, (x) - u_{1,2}\, (x) \tag{9}$$

o ancora:

$$\mathrm{curl}\, \mathbf{u} = \begin{pmatrix} \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{pmatrix} \tag{10}$$

■ I campi tensoriali

Sia **S** un campo tensoriale su B, e si supponga che esso sia differenziabile in x ∈ B, assieme al suo trasposto S^T.
1.) La *divergenza* di **S** nel punto x, div S(x), e' definito come l'unico vettore per cui:

$$[\mathrm{div}\, S\, (x)] \cdot a = \mathrm{div}[S^T\, (x)\, a] \tag{11}$$

per ogni vettore a. Indicialmente si ha:

$$[\mathrm{div}\, S\, (x)]_i = S_{ij,j}\, (x) \tag{12}$$

o, per esteso:

$$[\mathrm{div}\, S\, (x)]_1 = \frac{\partial S_{11}}{\partial x_1} + \frac{\partial S_{12}}{\partial x_2} + \frac{\partial S_{13}}{\partial x_3} \tag{13}$$

$$[\mathrm{div}\, S\, (x)]_2 = \frac{\partial S_{21}}{\partial x_1} + \frac{\partial S_{22}}{\partial x_2} + \frac{\partial S_{23}}{\partial x_3} \tag{14}$$

$$[\mathrm{div}\, S\, (x)]_3 = \frac{\partial S_{31}}{\partial x_1} + \frac{\partial S_{32}}{\partial x_2} + \frac{\partial S_{33}}{\partial x_3} \tag{15}$$

$$\mathrm{div}\, \mathbf{S} = \begin{pmatrix} \frac{\partial S_{11}}{\partial x_1} + \frac{\partial S_{12}}{\partial x_2} + \frac{\partial S_{13}}{\partial x_3} \\ \frac{\partial S_{21}}{\partial x_1} + \frac{\partial S_{22}}{\partial x_2} + \frac{\partial S_{23}}{\partial x_3} \\ \frac{\partial S_{31}}{\partial x_1} + \frac{\partial S_{32}}{\partial x_2} + \frac{\partial S_{33}}{\partial x_3} \end{pmatrix} \tag{16}$$

2.) Il *rotore* di S nel punto x, curl S(x), e' definito come l'unico tensore per cui si abbia:

$$[\mathrm{rot}\, S\, (x)] \cdot a = \mathrm{rot}[S^T\, (x)\, a] \tag{17}$$

per ogni vettore a. Indicialmente si ha:

$$[\mathrm{rot}\, S\, (x)]_{ij} = \epsilon_{ipk}\, S_{jk,p}\, (x) \tag{18}$$

o, per esteso:

$$\begin{aligned}
(\mathrm{rot}S)_{11} &= \epsilon_{1\,pk}\frac{\partial S_{1\,k}}{\partial x_p} = \epsilon_{123}\frac{\partial S_{13}}{\partial x_2} + \epsilon_{132}\frac{\partial S_{12}}{\partial x_3} = \frac{\partial S_{13}}{\partial x_2} - \frac{\partial S_{12}}{\partial x_3}\\
(\mathrm{rot}S)_{12} &= \epsilon_{1\,pk}\frac{\partial S_{2\,k}}{\partial x_p} = \epsilon_{123}\frac{\partial S_{23}}{\partial x_2} + \epsilon_{132}\frac{\partial S_{22}}{\partial x_3} = \frac{\partial S_{23}}{\partial x_2} - \frac{\partial S_{22}}{\partial x_3}\\
(\mathrm{rot}S)_{13} &= \epsilon_{1\,pk}\frac{\partial S_{3\,k}}{\partial x_p} = \epsilon_{123}\frac{\partial S_{33}}{\partial x_2} + \epsilon_{132}\frac{\partial S_{32}}{\partial x_3} = \frac{\partial S_{33}}{\partial x_2} - \frac{\partial S_{32}}{\partial x_3}
\end{aligned} \tag{19}$$

$$\begin{aligned}
(\mathrm{rot}S)_{21} &= \epsilon_{2\,pk}\frac{\partial S_{1\,k}}{\partial x_p} = \epsilon_{213}\frac{\partial S_{13}}{\partial x_1} + \epsilon_{231}\frac{\partial S_{11}}{\partial x_3} = \frac{\partial S_{11}}{\partial x_3} - \frac{\partial S_{13}}{\partial x_1}\\
(\mathrm{rot}S)_{22} &= \epsilon_{2\,pk}\frac{\partial S_{2\,k}}{\partial x_p} = \epsilon_{213}\frac{\partial S_{23}}{\partial x_1} + \epsilon_{231}\frac{\partial S_{21}}{\partial x_3} = \frac{\partial S_{21}}{\partial x_3} - \frac{\partial S_{23}}{\partial x_1}\\
(\mathrm{rot}S)_{23} &= \epsilon_{2\,pk}\frac{\partial S_{3\,k}}{\partial x_p} = \epsilon_{213}\frac{\partial S_{33}}{\partial x_1} + \epsilon_{231}\frac{\partial S_{31}}{\partial x_3} = \frac{\partial S_{31}}{\partial x_3} - \frac{\partial S_{33}}{\partial x_1}
\end{aligned} \tag{20}$$

$$\begin{aligned}
(\mathrm{rot}S)_{31} &= \epsilon_{3\,pk}\frac{\partial S_{1\,k}}{\partial x_p} = \epsilon_{312}\frac{\partial S_{12}}{\partial x_1} + \epsilon_{321}\frac{\partial S_{11}}{\partial x_2} = \frac{\partial S_{12}}{\partial x_1} - \frac{\partial S_{11}}{\partial x_2}\\
(\mathrm{rot}S)_{32} &= \epsilon_{3\,pk}\frac{\partial S_{2\,k}}{\partial x_p} = \epsilon_{312}\frac{\partial S_{22}}{\partial x_1} + \epsilon_{321}\frac{\partial S_{21}}{\partial x_2} = \frac{\partial S_{22}}{\partial x_1} - \frac{\partial S_{21}}{\partial x_2}\\
(\mathrm{rot}S)_{33} &= \epsilon_{3\,pk}\frac{\partial S_{3\,k}}{\partial x_p} = \epsilon_{312}\frac{\partial S_{32}}{\partial x_1} + \epsilon_{321}\frac{\partial S_{31}}{\partial x_2} = \frac{\partial S_{32}}{\partial x_1} - \frac{\partial S_{31}}{\partial x_2}
\end{aligned} \tag{21}$$

Matricialmente, e' quindi:

$$\mathrm{rot}\mathbf{S} = \begin{pmatrix} \frac{\partial S_{13}}{\partial x_2} - \frac{\partial S_{12}}{\partial x_3} & \frac{\partial S_{23}}{\partial x_2} - \frac{\partial S_{22}}{\partial x_3} & \frac{\partial S_{33}}{\partial x_2} - \frac{\partial S_{32}}{\partial x_3} \\ \frac{\partial S_{11}}{\partial x_3} - \frac{\partial S_{13}}{\partial x_1} & \frac{\partial S_{21}}{\partial x_3} - \frac{\partial S_{23}}{\partial x_1} & \frac{\partial S_{31}}{\partial x_3} - \frac{\partial S_{33}}{\partial x_1} \\ \frac{\partial S_{12}}{\partial x_1} - \frac{\partial S_{11}}{\partial x_2} & \frac{\partial S_{22}}{\partial x_1} - \frac{\partial S_{21}}{\partial x_2} & \frac{\partial S_{32}}{\partial x_1} - \frac{\partial S_{31}}{\partial x_2} \end{pmatrix} \tag{22}$$

```
D1 = Table[0, {i, 1, 3}, {j, 1, 3}];

Do[
 D1[[i, j]] = Sum[Signature[{i, p, k}] u[j, k, p], {p, 1, 3}, {k, 1, 3}],
 {i, 1, 3}, {j, 1, 3}]

D1
```

```
{{-u[1, 2, 3] + u[1, 3, 2],
  -u[2, 2, 3] + u[2, 3, 2], -u[3, 2, 3] + u[3, 3, 2]},
 {u[1, 1, 3] - u[1, 3, 1], u[2, 1, 3] - u[2, 3, 1], u[3, 1, 3] - u[3, 3, 1]},
 {-u[1, 1, 2] + u[1, 2, 1],
  -u[2, 1, 2] + u[2, 2, 1], -u[3, 1, 2] + u[3, 2, 1]}}
```

■ Il Laplaciano

Sia ϕ un campo scalare, e si supponga che il suo gradiente $\nabla\phi(x)$ sia differenziabile nel punto x. Il Laplaciano di ϕ in x si definisce come:

$$\Delta\phi\,(x) = \mathrm{div}\,\nabla\phi\,(x) \tag{23}$$

ossia, indicialmente:

$$\Delta\phi\,(x) = \phi_{,ii}\,(x) = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2} \qquad (24)$$

Analogamente, il Laplaciano di un campo vettoriale u e' definito come un ulteriore vettore:

$$\Delta u\,(x) = \mathrm{div}\,\nabla u\,(x) \qquad (25)$$

ed indicialmente:

$$[\Delta u\,(x)]_i = u_{i,jj}\,(x) \qquad (26)$$

e per esteso:

$$[\Delta u\,(x)]_1 = \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} \qquad (27)$$

$$[\Delta u\,(x)]_2 = \frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2} \qquad (28)$$

$$[\Delta u\,(x)]_3 = \frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_3^2} \qquad (29)$$

Infine, il laplaciano di un tensore S e' definito come l'unico tensore con la proprieta':

$$[\Delta S\,(x)]\,a = \Delta[S\,(x)\,a] \qquad (30)$$

per ogni vettore a. Indicialmente si ha:

$$[\Delta S\,(x)]_{ij} = S_{ij,kk}\,(x) \qquad (31)$$

Alcuni risultati notevoli

- **1. div(curl S)^T = 0**

Assegnato il rotore di S:

$$\mathrm{curl}\,\mathbf{S} = \begin{pmatrix} \frac{\partial S_{13}}{\partial x_2} - \frac{\partial S_{12}}{\partial x_3} & \frac{\partial S_{23}}{\partial x_2} - \frac{\partial S_{22}}{\partial x_3} & \frac{\partial S_{33}}{\partial x_2} - \frac{\partial S_{32}}{\partial x_3} \\ \frac{\partial S_{11}}{\partial x_3} - \frac{\partial S_{13}}{\partial x_1} & \frac{\partial S_{21}}{\partial x_3} - \frac{\partial S_{23}}{\partial x_1} & \frac{\partial S_{31}}{\partial x_3} - \frac{\partial S_{33}}{\partial x_1} \\ \frac{\partial S_{12}}{\partial x_1} - \frac{\partial S_{11}}{\partial x_2} & \frac{\partial S_{22}}{\partial x_1} - \frac{\partial S_{21}}{\partial x_2} & \frac{\partial S_{32}}{\partial x_1} - \frac{\partial S_{31}}{\partial x_2} \end{pmatrix} \qquad (32)$$

e' immediato dimostrare che la trasposta di questa matrice ha divergenza nulla: div (curl S)^T = 0:

Sia infatti T = curl S, e si calcolino le tre componenti della divergenza di T^T :

$$D_1 = \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{21}}{\partial x_2} + \frac{\partial T_{31}}{\partial x_3} = \frac{\partial S_{13}}{\partial x_1\,\partial x_2} - \frac{\partial S_{12}}{\partial x_1\,\partial x_3} + \frac{\partial S_{11}}{\partial x_2\,\partial x_3} - \frac{\partial S_{13}}{\partial x_2\,\partial x_1} + \frac{\partial S_{12}}{\partial x_3\,\partial x_1} - \frac{\partial S_{11}}{\partial x_3\,\partial x_2} = 0 \qquad (33)$$

$$D_2 = \frac{\partial T_{12}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{32}}{\partial x_3} = \frac{\partial S_{23}}{\partial x_1 \partial x_2} - \frac{\partial S_{22}}{\partial x_1 \partial x_3} + \frac{\partial S_{21}}{\partial x_2 \partial x_3} - \frac{\partial S_{23}}{\partial x_2 \partial x_1} + \frac{\partial S_{22}}{\partial x_3 \partial x_1} - \frac{\partial S_{21}}{\partial x_3 \partial x_2} = 0 \tag{34}$$

$$D_3 = \frac{\partial T_{13}}{\partial x_1} + \frac{\partial T_{23}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} = \frac{\partial S_{33}}{\partial x_1 \partial x_2} - \frac{\partial S_{32}}{\partial x_1 \partial x_3} + \frac{\partial S_{31}}{\partial x_2 \partial x_3} - \frac{\partial S_{33}}{\partial x_2 \partial x_1} + \frac{\partial S_{32}}{\partial x_3 \partial x_1} - \frac{\partial S_{31}}{\partial x_3 \partial x_2} = 0 \tag{35}$$

- **2. div curl S = curl div S^T**

Assegnato il rotore di S:

$$\text{curl}\, S = \begin{pmatrix} \frac{\partial S_{13}}{\partial x_2} - \frac{\partial S_{12}}{\partial x_3} & \frac{\partial S_{23}}{\partial x_2} - \frac{\partial S_{22}}{\partial x_3} & \frac{\partial S_{33}}{\partial x_2} - \frac{\partial S_{32}}{\partial x_3} \\ \frac{\partial S_{11}}{\partial x_3} - \frac{\partial S_{13}}{\partial x_1} & \frac{\partial S_{21}}{\partial x_3} - \frac{\partial S_{23}}{\partial x_1} & \frac{\partial S_{31}}{\partial x_3} - \frac{\partial S_{33}}{\partial x_1} \\ \frac{\partial S_{12}}{\partial x_1} - \frac{\partial S_{11}}{\partial x_2} & \frac{\partial S_{22}}{\partial x_1} - \frac{\partial S_{21}}{\partial x_2} & \frac{\partial S_{32}}{\partial x_1} - \frac{\partial S_{31}}{\partial x_2} \end{pmatrix} \tag{36}$$

se ne calcoli la divergenza:

$$D_1 = \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} = \frac{\partial S_{13}}{\partial x_1 \partial x_2} - \frac{\partial S_{12}}{\partial x_1 \partial x_3} + \frac{\partial S_{23}}{\partial x_2 \partial x_2} - \frac{\partial S_{22}}{\partial x_2 \partial x_3} + \frac{\partial S_{33}}{\partial x_3 \partial x_2} - \frac{\partial S_{32}}{\partial x_3 \partial x_3} \tag{37}$$

$$D_2 = \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} = \frac{\partial S_{11}}{\partial x_1 \partial x_3} - \frac{\partial S_{13}}{\partial x_1 \partial x_1} + \frac{\partial S_{21}}{\partial x_2 \partial x_3} - \frac{\partial S_{23}}{\partial x_2 \partial x_1} + \frac{\partial S_{31}}{\partial x_3 \partial x_3} - \frac{\partial S_{33}}{\partial x_3 \partial x_1} \tag{38}$$

$$D_3 = \frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} = \frac{\partial S_{12}}{\partial x_1 \partial x_1} - \frac{\partial S_{11}}{\partial x_1 \partial x_2} + \frac{\partial S_{22}}{\partial x_2 \partial x_1} - \frac{\partial S_{21}}{\partial x_2 \partial x_2} + \frac{\partial S_{32}}{\partial x_3 \partial x_1} - \frac{\partial S_{31}}{\partial x_3 \partial x_2} \tag{39}$$

D'altro canto, si calcoli la divergenza di S^T, ottenendo il vettore **d** = div S^T:

$$d_1 = \frac{\partial S_{11}}{\partial x_1} + \frac{\partial S_{21}}{\partial x_2} + \frac{\partial S_{31}}{\partial x_3} \tag{40}$$

$$d_2 = \frac{\partial S_{12}}{\partial x_1} + \frac{\partial S_{22}}{\partial x_2} + \frac{\partial S_{32}}{\partial x_3} \tag{41}$$

$$d_3 = \frac{\partial S_{13}}{\partial x_1} + \frac{\partial S_{23}}{\partial x_2} + \frac{\partial S_{33}}{\partial x_3} \tag{42}$$

Il rotore di **d** e' ora ottenibile come:

$$(\text{rot}\, d)_1 = \frac{\partial d_3}{\partial x_2} - \frac{\partial d_2}{\partial x_3} = \frac{\partial S_{13}}{\partial x_1 \partial x_2} + \frac{\partial S_{23}}{\partial x_2 \partial x_2} + \frac{\partial S_{33}}{\partial x_3 \partial x_2} - \frac{\partial S_{12}}{\partial x_1 \partial x_3} - \frac{\partial S_{22}}{\partial x_2 \partial x_3} - \frac{\partial S_{32}}{\partial x_3 \partial x_3} \tag{43}$$

$$(\mathrm{rot}\, d)_2 = \frac{\partial d_1}{\partial x_3} - \frac{\partial d_3}{\partial x_1} = \frac{\partial S_{11}}{\partial x_1 \partial x_3} + \frac{\partial S_{21}}{\partial x_2 \partial x_3} + \frac{\partial S_{31}}{\partial x_3 \partial x_3} - \frac{\partial S_{13}}{\partial x_1 \partial x_1} - \frac{\partial S_{23}}{\partial x_2 \partial x_1} - \frac{\partial S_{33}}{\partial x_3 \partial x_1} \tag{44}$$

$$(\mathrm{rot}\, d)_3 = \frac{\partial d_2}{\partial x_1} - \frac{\partial d_1}{\partial x_2} = \frac{\partial S_{12}}{\partial x_1 \partial x_1} + \frac{\partial S_{22}}{\partial x_2 \partial x_1} + \frac{\partial S_{32}}{\partial x_3 \partial x_1} - \frac{\partial S_{11}}{\partial x_1 \partial x_2} - \frac{\partial S_{21}}{\partial x_2 \partial x_2} - \frac{\partial S_{31}}{\partial x_3 \partial x_2} \tag{45}$$

ottenendo l'uguaglianza:

$$\mathrm{div}\,\mathrm{curl}\, S = \mathrm{curl}\,\mathrm{div}\, S^T \tag{46}$$

- **3. (curl curl $S)^T$ = curl curl S^T**

Si vuol dimostrare ora che il rotore del rotore di S, ed il rotore del rotore di S^T sono una la trasposta dell'altra. A tal fine si calcoli il rotore del rotore di S, in base alla formula:

$$A_{ij} = \mathrm{curl}\,\mathrm{curl}\, S = \epsilon_{imn}\, \epsilon_{jpq} \frac{\partial^2 S_{mp}}{\partial x_n \partial x_q} \tag{47}$$

ottenendo:

$$A_{11} = \frac{\partial^2 S_{22}}{\partial x_3 \partial x_3} - \frac{\partial^2 S_{23}}{\partial x_3 \partial x_2} - \frac{\partial^2 S_{32}}{\partial x_2 \partial x_3} + \frac{\partial^2 S_{33}}{\partial x_2 \partial x_2} \tag{48}$$

$$A_{12} = -\frac{\partial^2 S_{21}}{\partial x_3 \partial x_3} + \frac{\partial^2 S_{23}}{\partial x_3 \partial x_1} + \frac{\partial^2 S_{31}}{\partial x_2 \partial x_3} - \frac{\partial^2 S_{33}}{\partial x_2 \partial x_1} \tag{49}$$

$$A_{13} = \frac{\partial^2 S_{21}}{\partial x_3 \partial x_2} - \frac{\partial^2 S_{22}}{\partial x_3 \partial x_1} - \frac{\partial^2 S_{31}}{\partial x_2 \partial x_2} + \frac{\partial^2 S_{32}}{\partial x_2 \partial x_1} \tag{50}$$

$$A_{21} = -\frac{\partial^2 S_{12}}{\partial x_3 \partial x_3} + \frac{\partial^2 S_{13}}{\partial x_3 \partial x_2} + \frac{\partial^2 S_{32}}{\partial x_1 \partial x_3} - \frac{\partial^2 S_{33}}{\partial x_2 \partial x_1} \tag{51}$$

$$A_{22} = \frac{\partial^2 S_{11}}{\partial x_3 \partial x_3} - \frac{\partial^2 S_{13}}{\partial x_3 \partial x_1} - \frac{\partial^2 S_{31}}{\partial x_1 \partial x_3} + \frac{\partial^2 S_{33}}{\partial x_1 \partial x_1} \tag{52}$$

$$A_{23} = -\frac{\partial^2 S_{11}}{\partial x_3 \partial x_2} + \frac{\partial^2 S_{12}}{\partial x_3 \partial x_1} + \frac{\partial^2 S_{31}}{\partial x_1 \partial x_2} - \frac{\partial^2 S_{32}}{\partial x_1 \partial x_1} \tag{53}$$

$$A_{31} = \frac{\partial^2 S_{12}}{\partial x_2 \partial x_3} - \frac{\partial^2 S_{13}}{\partial x_2 \partial x_2} - \frac{\partial^2 S_{22}}{\partial x_1 \partial x_3} + \frac{\partial^2 S_{23}}{\partial x_1 \partial x_2} \tag{54}$$

$$A_{23} = -\frac{\partial^2 S_{11}}{\partial x_3 \partial x_2} + \frac{\partial^2 S_{13}}{\partial x_2 \partial x_1} + \frac{\partial^2 S_{21}}{\partial x_1 \partial x_3} - \frac{\partial^2 S_{23}}{\partial x_1 \partial x_1} \tag{55}$$

$$A_{33} = \frac{\partial^2 S_{11}}{\partial x_2 \partial x_2} - \frac{\partial^2 S_{12}}{\partial x_2 \partial x_1} - \frac{\partial^2 S_{21}}{\partial x_1 \partial x_2} + \frac{\partial^2 S_{22}}{\partial x_1 \partial x_1} \tag{56}$$

ed analogamente, si calcoli il rotore del rotore della trasposta di S:

$$B_{ij} = \mathrm{curl}\,\mathrm{curl}\, S^T = \epsilon_{imn}\, \epsilon_{jpq} \frac{\partial^2 S_{pm}}{\partial x_n \partial x_q} \tag{57}$$

Si ottiene quindi:

$$(\text{curl curl } S)^T = \text{curl curl } S^T \tag{58}$$

Funzioni Mathematica

■ Il rotore di un vettore

Il rotore di un vettore **u** si scrive:

$$[\text{curl } \mathbf{u}(x)]_i = \epsilon_{ijk} u_{k,j}(x) \tag{59}$$

Il tensore di Ricci, in *Mathematica*, assume il ruolo di Signature[i,j,k], e, per semplicita', si indica con u[i,j] la derivata di u_i rispetto ad x_j, sicche' la formula per il calcolo del rotore di un vettore puo' scriversi come:

```
Curlu = Table[0, {i, 1, 3}];

Do[
 Curlu[[i]] = Sum[Signature[{i, j, k}] u[k, j], {j, 1, 3}, {k, 1, 3}],
 {i, 1, 3}]

Curlu // MatrixForm
```

$$\begin{pmatrix} -u[2,3]+u[3,2] \\ u[1,3]-u[3,1] \\ -u[1,2]+u[2,1] \end{pmatrix}$$

■ Il rotore di un tensore

Il rotore di un tensore **S** si scrive:

$$[\text{rot } S(x)]_{ij} = \epsilon_{ipk} S_{jk,p}(x) \tag{60}$$

Per semplicita', si indica con S[i,j,k] la derivata di S_{ij} rispetto ad x_k, sicche' la formula per il calcolo del rotore di un tensore puo' scriversi come:

```
CurlS = Table[0, {i, 1, 3}, {j, 1, 3}];

Do[
 CurlS[[i, j]] =
  Sum[Signature[{i, p, k}] S[j, k, p], {p, 1, 3}, {k, 1, 3}],
 {i, 1, 3}, {j, 1, 3}]

CurlS // MatrixForm
```

$$\begin{pmatrix} -S[1,2,3]+S[1,3,2] & -S[2,2,3]+S[2,3,2] & -S[3,2,3]+S[3,3,2] \\ S[1,1,3]-S[1,3,1] & S[2,1,3]-S[2,3,1] & S[3,1,3]-S[3,3,1] \\ -S[1,1,2]+S[1,2,1] & -S[2,1,2]+S[2,2,1] & -S[3,1,2]+S[3,2,1] \end{pmatrix}$$

■ Il rotore del rotore di un tensore

Il rotore del rotore di un tensore **S** si scrive:

$$A_{ij} = \text{curl curl } S = \epsilon_{imn}\,\epsilon_{jpq}\,\frac{\partial^2 S_{mp}}{\partial x_n\,\partial x_q} \qquad (61)$$

Per semplicita', si indica con S[i,j,k,l] la derivata seconda di S_{ij} rispetto ad x_k ed x_l, sicche' la formula per il calcolo del rotore di un tensore puo' scriversi come:

```
A = Table[0, {i, 1, 3}, {j, 1, 3}];

Do[
  A[[i, j]] = Sum[Signature[{i, m, n}] Signature[{j, p, q}] S[m, p, n, q],
    {m, 1, 3}, {n, 1, 3}, {p, 1, 3}, {q, 1, 3}],
  {i, 1, 3}, {j, 1, 3}];

A

{{S[2, 2, 3, 3] - S[2, 3, 3, 2] - S[3, 2, 2, 3] + S[3, 3, 2, 2],
  -S[2, 1, 3, 3] + S[2, 3, 3, 1] + S[3, 1, 2, 3] - S[3, 3, 2, 1],
  S[2, 1, 3, 2] - S[2, 2, 3, 1] - S[3, 1, 2, 2] + S[3, 2, 2, 1]},
 {-S[1, 2, 3, 3] + S[1, 3, 3, 2] + S[3, 2, 1, 3] - S[3, 3, 1, 2],
  S[1, 1, 3, 3] - S[1, 3, 3, 1] - S[3, 1, 1, 3] + S[3, 3, 1, 1],
  -S[1, 1, 3, 2] + S[1, 2, 3, 1] + S[3, 1, 1, 2] - S[3, 2, 1, 1]},
 {S[1, 2, 2, 3] - S[1, 3, 2, 2] - S[2, 2, 1, 3] + S[2, 3, 1, 2],
  -S[1, 1, 2, 3] + S[1, 3, 2, 1] + S[2, 1, 1, 3] - S[2, 3, 1, 1],
  S[1, 1, 2, 2] - S[1, 2, 2, 1] - S[2, 1, 1, 2] + S[2, 2, 1, 1]}}
```

■ Il rotore del rotore della trasposta di S

Il rotore del rotore della trasposta di S si scrive:

$$B_{ij} = \text{curl curl } S^T = \epsilon_{imn}\,\epsilon_{jpq}\,\frac{\partial^2 S_{pm}}{\partial x_n\,\partial x_q} \qquad (62)$$

e quindi *Mathematica* fornisce subito:

```
B = Table[0, {i, 1, 3}, {j, 1, 3}];

Do[
 B[[i, j]] = Sum[Signature[{i, m, n}] Signature[{j, p, q}] S[p, m, n, q],
   {m, 1, 3}, {n, 1, 3}, {p, 1, 3}, {q, 1, 3}],
 {i, 1, 3}, {j, 1, 3}]
```

■ Verifica : $(\text{curl curl } S)^T = \text{curl curl } S^T$

Deve essere:

```
Simplify[Transpose[A] - B]
```

```
{{S[2, 3, 2, 3] - S[2, 3, 3, 2] - S[3, 2, 2, 3] + S[3, 2, 3, 2],
  -S[1, 3, 2, 3] + S[1, 3, 3, 2] + S[3, 2, 1, 3] - S[3, 2, 3, 1] -
   S[3, 3, 1, 2] + S[3, 3, 2, 1], S[1, 2, 2, 3] - S[1, 2, 3, 2] -
   S[2, 2, 1, 3] + S[2, 2, 3, 1] + S[2, 3, 1, 2] - S[2, 3, 2, 1]},
 {-S[2, 3, 1, 3] + S[2, 3, 3, 1] + S[3, 1, 2, 3] -
   S[3, 1, 3, 2] + S[3, 3, 1, 2] - S[3, 3, 2, 1],
  S[1, 3, 1, 3] - S[1, 3, 3, 1] - S[3, 1, 1, 3] + S[3, 1, 3, 1],
  -S[1, 1, 2, 3] + S[1, 1, 3, 2] - S[1, 3, 1, 2] +
   S[1, 3, 2, 1] + S[2, 1, 1, 3] - S[2, 1, 3, 1]},
 {-S[2, 1, 2, 3] + S[2, 1, 3, 2] + S[2, 2, 1, 3] - S[2, 2, 3, 1] -
   S[3, 2, 1, 2] + S[3, 2, 2, 1], S[1, 1, 2, 3] - S[1, 1, 3, 2] -
   S[1, 2, 1, 3] + S[1, 2, 3, 1] + S[3, 1, 1, 2] - S[3, 1, 2, 1],
  S[1, 2, 1, 2] - S[1, 2, 2, 1] - S[2, 1, 1, 2] + S[2, 1, 2, 1]}}
```

e poiche', per il teorema di Schwartz, S[i,j,k,l] = S[i,j,l,k], si ha A^T = B