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THREE SUBROUTINES FOR THE ANALYSIS OF SUSPENSION BRIDGES

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Abstract—A new procedure for analysis of suspension bridges has been recently presented (Franciosi and Franciosi, *Comput. Struct.* **26,** 499–512, 1987), in which the so-called cell method was employed in order to obtain static and dynamic response of a one-span suspension bridge.

In the present paper three efficient subroutines are introduced, which allow us to calculate the strain energy matrix, Lagrangian mass matrix and participation factors of each vibration mode. Every matrical operation is avoided, so that the proposed method is very fast and manageable. Both the matrices can be immediately built, and a standard eigenvalue package will furnish eigenvalues and eigenvectors. Then, the third subroutine calculates the participation factors of the modes, for synchronous and asynchronous earthquakes.

A numerical example shows that the cell method leads to good approximations, even when the discretization is very rough.

1. INTRODUCTION

Consider a one-span suspension bridge, whose span is l, whose sag is f, subjected to a distributed dead load g. The cell method discretizes the structure in such a way that the following free vibration equation is obtained:

$$(K + GB)c - \omega^2 Mc = 0 \tag{1}$$

where

$$G = \frac{gl^3}{8f(n+1)}$$

and n is the number of elastic cells in which the strain energy is supposed to be concentrated.

The strain energy matrix, K, the stiffening matrix, B, and the Lagrangian matrix of the masses, M, can be calculated as in [1], but the matrical formulation is clearly not very suitable for the computer. For the sake of clarity, we report here some interesting formulae.

The matrix K is given in Table 1, where k is the array of the concentrated dibilities, and c_D^0 is the horizontal displacement of the left abutment due to a force $H = 1$, and in presence of fixed Lagrangian coordinates. (See [1, formulae 20].)

The matrix B is simply given by $B_{ii} = 2$ and $B_{ij} = 1$ if $i \neq j$. Finally, the Lagrangian matrix of the masses can be obtained from the triple matrical product

$$\tilde{M} = V^T MV, \tag{2}$$

where M is the (diagonal) matrix of the concentrated masses, and V is the diplacement matrix of these masses. V is given by

$$-\frac{l}{n+1} \quad \text{if} \quad 2 \leqslant j \leqslant i-1$$

$$V_{ij} = 0 \quad \text{if} \quad i \leqslant j \leqslant n+1. \tag{3}$$

If the three matrices are obtained, then the eigenvalue problem (1) can be solved, by means of a usual eigensolution package, and the frequencies ω_i^2 can be detected, together with their free vibration mode e_i.

Modal superposition analysis is then used in order to obtain the seismic response of the bridge. In [1] the earthquake was assimilated to a sinusoidal displacement of the abutments, whose frequency is ω_s^2. In this case, the participation factors of each mode are given by

$$p_{ai} = \frac{e_i^T b}{\omega_s^2 e_i^T Me_i} \tag{4}$$

if only one of the abutments moves, or by

$$p_{si} = \frac{e_i^T V^T m}{e_i^T Me} \tag{5}$$

if both the abutments move in a synchronous way. The array m contains the concentrated masses, while b is given by:

$$b_i = \frac{8f}{(n+1)^2 c_D^0}(1 + n - i).$$

2. THE SUBROUTINES

The three subroutines are given in the Appendix. The first one allows us to obtain (K + GB), according to the scheme in Table 1. Intput data are indicated in the listing, and the procedure is quite simple: for each row i the following elements are calculated:

—the main diagonal element (row 350)
—the element $(i+1, i)$ (row 360)
—the elements (j, i) for j from $i+2$ to $n-1$ (rows 370–390)
—the element (n, i) (row 400).

Table 1. Strain energy matrix

$i = j$	
$k_{ii} = k_i + k_{i+1} + k_{n+1} + \dfrac{64f^2}{(n+1)^4 c_D^0}(1+n-i)^2$	if $i < n$
$k_{nn} = k_n + 4k_{n+1} + \dfrac{64f^2}{(n+1)^4 c_D^0}$	if $i = n$
$i < j < n$	
$k_{ij} + k_{n+1} + \dfrac{64f^2}{(n+1)^4 c_D^0}(1+n-i)(1+n-j)$	if $j > i+1$
$k_{i,i+1} = -k_{i+1} + k_{n+1} + \dfrac{64f^2}{(n+1)^4 c_D^0}(1+n-i)(n-i)$	if $j = i+1$
$j = n$	
$k_{in} = 2k_{n+1} + \dfrac{64f^2}{(n+1)^4 c_D^0}(1+n-i)$	if $i < n-1$
$k_{n-1,n} = -k_n + 2k_{n+1} + 2\dfrac{64f^2}{(n+1)^4 c_D^0}$	if $i = n-1$

Finally, the elements $(n, n-1)$ and (n, n) are calculated with the rows 420–430. The rows 490–570 add the second order effects.

The second subroutine calculates the Lagrangian matrix of the masses, by means of triple matrical product (2). It is easy to see that the particular structure of the matrix V, and the diagonal form of the matrix M reduce the routine to the loop 310–360, and every matrical operation can be avoided. This subroutine assumes a constant mass distribution, but it is evident that a slightly modified routine can handle a general mass distribution.

With the aid of these two subroutines an eigenvalue problem is defined, which can be solved by means of a usual eigensolution package. Natural periods and free vibration modes are thus obtained.

Finally, the third subroutine uses these results in order to detect the seismic response of the bridge to both synchronous and asynchronous earthquakes. Modal participation factors are calculated, according to eqns (4) and (5). First, the common denominator is obtained (rows 400–500), then two simple loops (rows 590–610 and 720–740) give the desired quantities. It is worth noting that even in this case the structure of the V matrix allows us to avoid the matrical products.

3. A NUMERICAL EXAMPLE

A one-span suspension bridge is considered, which was examined previously in [1]. Its geometrical data are reported in Table 2, and it is only necessary to add that the hangers' number is equal to 165. Therefore, it seems rather artificial to discretize the beam by introducing more than 165 elastic cells. In any case, we shall see that the cell method has convergence properties which allow us to drastically reduce the number of the Lagrangian coordinates. For example, in Table 3 the first five eigenvalues are reported, for various discretization degrees. It is easy to see that the introduction of 40 Lagrangian coordinates gives excellent results.

These eigenvalues were obtained by means of a routine which uses the Householder reduction, and the Sturm sequence properties [2], because only the first eigenvalues were required. In the last case

Table 2. Geometrical data of the proposed Messina suspension bridge

Description	Name	Value
Central span	l	3300 m
Lateral span	l_r	990 m
Sag of the cable	f	300 m
Distance between the cable and the bridge at the middle of the central span	c	12 m
Area of cable	A_c	5.856 m^2
Total area of the hangers	A_p	7.491 m^2
Young's modulus of the cable	E_c	18,000,000 tm^{-2}
Young's modulus of bridge and hangers	E_t	21,000,000 tm^{-2}
(Constant) moment of inertia of the bridge	I	6.019 m^4
Dead load per unit length	g	94 tm^{-1}

Table 3. The first five eigenvalues for the different discretrization degrees

	λ_1	λ_2	λ_3	λ_4	λ_5
$n = 3$	0.1309	0.2041	0.2852	—	—
$n = 5$	0.1473	0.2267	0.3573	0.4427	0.5537
$n = 10$	0.1572	0.2354	0.4085	0.5805	0.8632
$n = 20$	0.1603	0.2378	0.4257	0.6290	0.9812
$n = 40$	0.1612	0.2384	0.4305	0.6431	1.016
$n = 80$	0.1614	0.2385	0.4318	0.6469	1.026
$n = 165$	0.1614	0.2385	0.4318	0.6469	1.026

($n = 165$) the simultaneous inverse iteration method, as given in [3], is slightly more economical, while the Levit min–max method [4] seems to be rather unsatisfactory, at least in the eigenvectors calculation.

Bending moments and shear stresses due to an earthquake can be calculated by means of the modal superposition principle, according to [1]. If the earthquake period is far from the natural periods of the structure, then the bending moments are nearly constant along the beam, and consequently the shear stresses are almost zero. In Table 4 the bending moment at the middle point of the beam is given for different discretization degrees. It is necessary to obtain all the eigenvalues and eigenvectors, hence the CPU time increases considerably.

Fortunately, the convergence properties of the cell method lead to a very good result even if three Lagrangian coordinates are introduced. This means that the cell method can be implemented on a computer with very limited memory capacity, and the results will be satisfactory.

Table 4. Bending moment at the middle point of the beam, for different discretization degrees

n	3	5	10	20	40
M	410.1	398.0	391.5	390.1	389.8

Note: The period of the earthquake is 0.3 sec, hence the bridge is far from resonance.

The computation of all the eigenvalues and eigenvectors was conducted through the Wilkinson QL method [5] and through the classical Jacobi method [6], and they have proved to be almost equivalent. Of course, the other above mentioned methods are more expensive.

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APPENDIX

In this appendix the listing of the three subroutines is given. They are written in HP BASIC, and they were implemented on the HP 9807 Integral Personal Computer.

```
10 SUB "MatK" (k(),l,f,g,cd,n,a(,))
20 REM  *****************************************************************
30 REM  *                                                               *
40 REM  *                                                               *
50 REM  *              S U B R O U T I N E    MatK                      *
60 REM  *                                                               *
70 REM  *                                                               *
80 REM  *                                                               *
90 REM  *                     STRAIN ENERGY MATRIX                      *
100 REM *           TAKING INTO ACCOUNT SECOND ORDER EFFECTS            *
110 REM *                                                               *
120 REM *****************************************************************
130 OPTION BASE 1
140 REM -----------------------------------------------------------------
150 REM                V A R I A B L E S   I N D E X
160 REM------------------------------------------------------------------
170 REM
180 REM *  INPUT DATA:                                                  *
190 REM *              k(n+2) real    array of concentrated cedibilities *
200 REM *              l  real        span of the bridge                *
210 REM *              f  real        sag of the cable                  *
220 REM *              g  real        dead load for unit length         *
230 REM *              cd real        horizontal displacement of the    *
240 REM *                               left end due to H=1             *
250 REM *              n  integer     number of lagrangian coordinates  *
260 REM *                                                               *
```

```
270 REM ****************************************************************
280 REM *                                                              *
290 REM *  OUTPUT DATA :                                               *
300 REM *               a(n,n) real      strain energy matrix          *
310 REM *                                                              *
320 REM ****************************************************************
330 REM
340    FOR i=1 TO n-1
350         a(i,i)=k(i)+k(i+1)+k(n+1)+64*f^2/(n+1)^4/cd*(1+n-i)^2
360         a(i,i+1),a(i+1,i)=-k(i+1)+k(n+1)+64*f^2/(n+1)^4/cd*(1+n-i)*(n-i)
370              FOR j=i+2 TO n-1
380                   a(i,j),a(j,i)=k(n+1)+64*f^2/(n+1)^4/cd*(1+n-i)*(1+n-j)
390              NEXT j
400         a(i,n),a(n,i)=2*k(n+1)+64*f^2/(n+1)^4/cd*(1-i+n)
410    NEXT i
420  a(n,n-1),a(n-1,n)=-k(n)+2*k(n+1)+64*f^2/(n+1)^4/cd*2
430  a(n,n)=k(n)+4*k(n+1)+64*f^2/(n+1)^4/cd
440 REM
450 REM ----------------------------------------------------------------
460 REM              S E C O N D    O R D E R    E F F E C T S
470 REM ----------------------------------------------------------------
480 REM
490 const=g*l^3/8/f/(n+1)
500 FOR i=1 TO n
510 a(i,i)=a(i,i)+2*const
520 NEXT i
530 FOR i=2 TO n
540 FOR j=1 TO i-1
550 a(i,j),a(j,i)=a(i,j)+const
560 NEXT j
570 NEXT i
580 SUBEND
```

```
 10 SUB "MatM" (l,n,mass,b(,))
 30 REM  ****************************************************************
 40 REM  *                                                              *
 50 REM  *                                                              *
 60 REM  *              S U B R O U T I N E    MatM                     *
 70 REM  *                                                              *
 80 REM  *                                                              *
 90 REM  *                                                              *
 100 REM *               LAGRANGIAN MATRIX OF THE MASSES                *
 110 REM *                                                              *
 120 REM ****************************************************************
 130      OPTION BASE 1
 140 REM ----------------------------------------------------------------
 150 REM                V A R I A B L E S    I N D E X
 160 REM ----------------------------------------------------------------
 170 REM
 180 REM *  INPUT DATA :                                                *
 190 REM *               l     real       span of the bridge            *
 200 REM *               n     integer    number of lagrangian coordinates *
 210 REM *               mass  real       distributed masses on the bridge *
 220 REM *                                                              *
 230 REM ****************************************************************
 240 REM *                                                              *
 250 REM *  OUTPUT DATA :                                               *
 260 REM *                   b(n,n)     lagrangian matrix of the masses *
 270 REM *                                                              *
 280 REM ****************************************************************
 290 REM
 300 const=((-l)/(n+1))^2*mass
 310 FOR i=1 TO n
 320 FOR j=1 TO i
 340 b(i,j),b(j,i)=(n-i+1)*const
 350 NEXT j
 360 NEXT i
 370 SUBEND
```

```
 10 SUB "Particip" (n,nmodes,mass,l,b(,),vet(,),bsm(),om2s,ps(),pa())
 20 REM  ****************************************************************
 30 REM  *                                                              *
 40 REM  *                                                              *
 50 REM  *              S U B R O U T I N E    Particip                 *
 60 REM  *                                                              *
 70 REM  *                                                              *
 80 REM  *                                                              *
```

```
90 REM  *                    PARTICIPATION COEFFICIENTS FOR                    *
100 REM *              SYNCHRONOUS AND ASYNCHRONOUS EARTHQUAKES                *
110 REM *                                                                      *
120 REM ************************************************************************
130 REM
140 OPTION BASE 1
150 REM -----------------------------------------------------------------------
160 REM                 V A R I A B L E S   I N D E X
170 REM -----------------------------------------------------------------------
180 REM
190 REM ************************************************************************
200 REM *  INPUT DATA:                                                         *
210 REM *                   n            number of lagrangian coordinates      *
220 REM *                   nmodes       number of computed eigenvalues        *
230 REM *                   mass         distributed mass                      *
240 REM *                   l                 bridge span                      *
250 REM *                   b(n,n)            lagrangian matrix of masses      *
260 REM *                   vet(n,nmodes)     eigenvectors                     *
270 REM *                   bsm(n)            subsidiary array                 *
280 REM *                   om2s              earthquake frequency             *
290 REM *                                                                      *
300 REM ************************************************************************
310 REM *                                                                      *
320 REM *  OUTPUT DATA:                                                        *
330 REM *                   ps(nmodes)  particip. coeff. for sync. earth.      *
340 REM *                   pa(nmodes)  particip. coeff. for asyn. earth.      *
350 REM *                                                                      *
360 REM ************************************************************************
370 REM
380 REM
390 const=(-mass)*l/(n+1)
400 FOR i=1 TO nmodes
410  pl=0
420         FOR j=1 TO n
430         avl(j)=0
440            FOR k=1 TO n
450                avl(j)=avl(j)+b(j,k)*vet(k,i)
460            NEXT k
470         NEXT j
480  FOR j=1 TO n
490        pl=pl+avl(j)*vet(j,i)
500  NEXT j
510 REM
520 REM ************************************************************************
530 REM *                                                                      *
540 REM *          S I N C H R O N O U S    E A R T H Q U A K E                *
550 REM *                                                                      *
560 REM ************************************************************************
570 REM
580  ps(i)=0
590      FOR j=1 TO n
600        ps(i)=ps(i)+vet(j,i)*bsm(j)
610      NEXT j
620  ps(i)=ps(i)/pl/om2s
630 REM
640 REM
650 REM ************************************************************************
660 REM *                                                                      *
670 REM *        A S I N C H R O N O U S    E A R T H Q U A K E                *
680 REM *                                                                      *
690 REM ************************************************************************
700 REM
710       pa(i)=0
720         FOR j=1 TO n
730            pa(i)=pa(i)+vet(j,i)*const*(n-j+1)
740         NEXT j
750 pa(i)=pa(i)/pl
760 NEXT i
770 SUBEND
```