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THE OPTIMIZED RAYLEIGH METHOD AND *Mathematica* IN VIBRATIONS AND BUCKLING PROBLEMS

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In this paper some vibration and buckling problems are solved by means of the optimized Rayleigh and Timoshenko quotients. The use of the *Mathematica* symbolic language produces closer approximations than the usual ones, because two-parameter quotients can be employed.

1. INTRODUCTION

Since its introduction in 1870 [1], Rayleigh's quotient has been extensively used to give approximate values of natural frequencies and buckling loads for very many one-dimensional structures. An improved version of this quotient was proposed by the same author in 1894, [2], who used a trial function polynomial with a non-integer undetermined power (the so-called *non-integer power Rayleigh method*).

Although the method was successfully used by Stodola in 1927 [3], the intrinsic mathematical difficulties led to neglect of this powerful approach, which was recently rediscovered by Schmidt and Bert [4, 5], and consequently the non-integer power Rayleigh method is now known as the *optimized Rayleigh* method [6, 7]. More recently, it has been extended to cover two-dimensional problems by applying the Kantorovich method [8, 9].

Later on, a different implementation of the same quotient was proposed by Elishakoff [10], in which an undetermined multiplier rather than an undetermined power is used (*non-integer multiplier Rayleigh method*). It seems that this choice leads to simpler formulae, and even to more accurate results [11, 12].

In this paper the powerful *Mathematica* [13] symbolic language is used to obtain very accurate approximations to some vibrations and buckling problems. The use of symbolic software allowed the application for the first time—to the authors' knowledge—the non-integer multiplier Rayleigh method with *two* undetermined multipliers, and this approach probably yields the same accuracy as a four-term polynomial approximation with fixed exponents [14]. On the other hand, two exponential parameters were used by Grossi *et al*. [15].

Finally, an interesting application to vibration problems in the presence of axial compressive loads leads to a close approximation to the whole frequency-axial load curve, so enabling the critical load to be calculated even for pseudo-conservative systems [16].

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2. VIBRATION PROBLEMS

As a first example, consider a slender conical tapered bar with span L, in which the Young modulus E is supposed to be constant and the cross-sectional area and the mass per unit length are given by

$$A(X) = 2a_0X, \qquad m(X) = 2m_0X, \tag{1}$$

with $X = x/L$. The first axial natural frequency of this structure has been calculated by Bert [17] using the trial function

$$w(X) = C(X^n - 1). \tag{2}$$

The optimal value of n to minimize the Rayleigh quotient,

$$\bar{\omega}^2(n) = \int_0^1 EAw'^2\,\mathrm{d}X \bigg/ \int_0^1 mw^2\,\mathrm{d}X, \tag{3}$$

is equal to $\sqrt{2} \approx 1{\cdot}4142$, and the corresponding non-dimensional radian frequency,

$$\bar{\omega} = \sqrt{\bar{\omega}^2 m_0 L^2/Ea_0}, \tag{4}$$

is found to be equal to 2·4142.

If a non-integer multiplier Rayleigh approach is used, then one can use the trial function

$$w(X) = (X^2 - 1) + k(X^4 - 1), \tag{5}$$

since the boundary conditions

$$w(1) = 0, \qquad w'(0) = 0 \tag{6}$$

are both satisfied.

The following lines of *Mathematica* solve the problem:

```
v = (x^2 - 1) + k(x^4 - 1);
a = 2x;
m = 2x;
nray = Simplify[Integrate[a * D[v, x]^2, {x, 0, 1}];
dray = Simplify[Integrate[m * v^2, {x, 0, 1}];
ray = Simplify[nray/dray];
Simplify[Solve[D[ray, k] = = 0, k]]
ray/.%
```

Two values of the undetermined multiplier k are as obtained as

$$k_{1,2} = (-12 \pm \sqrt{34})/22 \tag{7}$$

and the corresponding values of the Rayleigh quotients are

$$\bar{\omega}_{1,2}^2 = \frac{20(3 \pm 4/11(\mp 12 + \sqrt{34}) + 3/242(\mp 12 + \sqrt{34})^2}{10 \pm 25/22(\mp 12 + \sqrt{34}) + 4/121(\mp 12 + \sqrt{34})^2}, \tag{8}$$

and, numerically,

$$\bar{\omega}_1 \approx 2{\cdot}40502. \tag{9}$$

This value is very close to the true result ($\bar{\omega}_1 = 2{\cdot}4048$ [18]), so that Elishakoff's statement [14]: "the approximation with an adjustable parameter entering linearly turned out to yield more straightforward results than the non-integer power version of Rayleigh" is confirmed.

The use of *Mathematica* allows the choice of the two-parameter trial function

$$w(X) = (X^2 - 1) + k(X^4 - 1) + k_1(X^6 - 1). \tag{10}$$

In this case it is necessary to solve the non-linear system:

$$\partial\bar{\omega}^2/\partial k = 0, \qquad \partial\bar{\omega}^2/\partial k_1 = 0, \tag{11}$$

and this is a formidable task, which can be tackled with the aid of symbolic languages.
The following lines of *Mathematica* solve the problem:

```
v = (x^2 - 1) + k(x^4 - 1) + k1(x^6 - 1);
a = 2x;
m = 2x;
nray = Simplify[Integrate[a * D[v, x]^2,{x, 0, 1}];
dray = Simplify[Integrate[m * v^2, {x, 0, 1}];
ray = Simplify[nray/dray];
Simplify[Solve[{D[ray, k] = = 0,  D[ray, k1] = = 0},{k, k1}]]
ray/.%
```

The optimized values of the two coefficients (k, k_1) are equal to:

$$(k, k_1)_1 = (-2{\cdot}44894633, 1{\cdot}52783972),$$
$$(k, k_1)_2 = (-0{\cdot}35414496, 0{\cdot}04715234),$$
$$(k, k_1)_3 = (-1{\cdot}43815986, 0{\cdot}58511943). \tag{12}$$

From the second set of coefficients

$$\bar{\omega}_1 \approx 2{\cdot}404825757, \tag{13}$$

which is almost coincident with the exact value.

Sometimes it is also useful to adopt a parametric version of the optimized Rayleigh method, in which one, or more, parameters are left unspecified.

Consider, for example, the cantilever beam of span L in Figure 1, with rectangular cross-section and linearly varying breadth. Both the area and the second moment of area of the cross section will vary according to the same law ($\zeta = z/L$),

$$I(\zeta) = I_0(1 - c\zeta), \qquad A(\zeta) = A_0(1 - c\zeta), \tag{14, 15}$$

and it is convenient to apply the optimization procedure without specifying the value of the parameter c. In this way, a single formula will be deduced, which is valid for complete and truncated beams (see reference [19]). If the following trial function is adopted,

$$v(\zeta) = \zeta^2 + k\zeta^3, \tag{16}$$

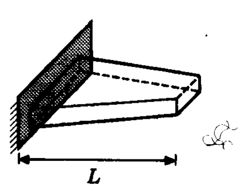

Figure 1. Cantilever beam with linearly varying breadth.

TABLE 1

Non-dimensional frequency for cantilever beam with varying cross-section: one-parameter approximation

	$I = I_0(1 - c\zeta)$			$I = I_0(1 - c\zeta)^3$	
c	k	$\bar{\omega}$	Exact [19]	k	$\bar{\omega}$
0	−0·38367	3·53273	3·5160	−0·38367	3·53273
0·1	−0·38172	3·64436	—	−0·37574	3·56519
0·2	−0·37948	3·77312	—	−0·36407	3·61056
0·3	−0·37691	3·92360	—	−0·34649	3·67087
0·4	−0·37395	4·10232	4·0970	−0·31931	3·74845
0·5	−0·37053	4·31882	4·3152	−0·27600	3·84587
0·6	−0·36662	4·58787	4·5853	−0·20455	3·96645
0·7	−0·36237	4·93370	4·9316	−0·08283	4·11671
0·8	−0·35848	5·39962	5·3976	0·12703	4·31677
0·9	−0·35791	6·07257	6·0704	0·46271	4·63618
1	−0·37400	7·15899	7·1565	0·77548	5·31874

then the quotient

$$\bar{\omega}^2(k) = \int_0^1 EIv''^2(\zeta)\,\mathrm{d}\zeta \Big/ \int_0^1 \rho A v^2(\zeta)\,\mathrm{d}\zeta \tag{17}$$

can be optimized with respect to k, and the following two values of the unknown multiplier are obtained,

$$k_{1,2} = (A \pm \sqrt{B})/C, \tag{18}$$

where

$$A = -768 + 1266c - 525c^2,$$
$$B = 3(53248 - 170112c + 203532c^2 - 108060c^3 + 21475c^4),$$
$$C = 30(32 - 53c + 22c^2), \tag{19}$$

and ρ is the mass density of the beam.

It is also common to deal with beams with varying height, where

$$I(\zeta) = I_0(1 - c\zeta)^3, \qquad A(\zeta) = A_0(1 - c\zeta). \tag{20, 21}$$

If the same trial function (16) is used, then the three coefficients in equation (18) are given

$$A = -7680 + 25380c - 31494c^2 + 17460c^3 - 3675c^4,$$
$$B = 3(5324800 - 33753600c + 94901680c^2 - 154384080c^3 + 158748012c^4$$
$$- 105529400c^5 + 44230940c^6 - 10670280c^7 + 1132275c^8),$$
$$C = 30(320 - 1050c + 1308c^2 - 733c^3 + 156c^4). \tag{22}$$

In Table 1 the first non-dimensional frequency is given, for both the variation laws, and a comparison is also made with the exact results given in reference [19]. The value $c = 0$ corresponds to a constant cross section, while $c = 1$ corresponds to a cantilever beam with a sharp end. The optimum value of the multiplier is also reported, and it is perhaps worth noting that the beam with linearly varying breadth is characterized by an almost constant t value.

The optimization method can also be employed to approximate higher eigenvalues, as illustrated by Laura and Cortinez [20] using a Galerkin approach and a one-parameter non-integer power Rayleigh method.

In the following a similar procedure will be employed in order to find close upper bounds to the first two frequencies for the tapered cantilever beam with linearly varying height. Moreover, a two-parameter non-integer multiplier Rayleigh method will be adopted, using a Ritz approach.

To this end, the following trial function is defined:

$$v(z) = (a_1 z^2 + a_2 z^3)(1 + tz + t_1 z^2). \tag{23}$$

The strain energy and the kinetic energy are readily calculated, and the following stiffness matrix and mass matrix are deduced:

$$\mathbf{k} = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}, \qquad \mathbf{m} = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}. \tag{24}$$

Here

$$\begin{aligned} k_{11} = {} & 8 - 12c + 8c^2 - 2c^3 + t(24 - 48c + 36c^2 - \tfrac{48}{5}c^3) + t^2(24 - 54c + \tfrac{216}{5}c^2 - 12c^3) \\ & + t_1(32 - 72c + \tfrac{288}{5}c^2 - 16c^3) + tt_1(72 - \tfrac{864}{5}c + 144c^2 - \tfrac{288}{7}c^3) \\ & + t_1^2(\tfrac{288}{5} - 144c + \tfrac{864}{7}c^2 - 36c^3), \end{aligned} \tag{25}$$

$$\begin{aligned} k_{12} = {} & 12 - 24c + 18c^2 - \tfrac{24}{5}c^3 + t(40 - 90c + 72c^2 - 20c^3) + t^2(36 - \tfrac{432}{5}c + 72c^2 - \tfrac{144}{7}c^3) \\ & + t_1(56 - \tfrac{672}{5}c + 112c^2 - 32c^3) + tt_1(\tfrac{528}{5} - 264c + \tfrac{1584}{7}c^2 - 66c^3) \\ & + t_1^2(80 - \tfrac{1440}{7}c + 180c^2 - \tfrac{160}{3}c^3), \end{aligned} \tag{26}$$

$$\begin{aligned} k_{22} = {} & 24 - 54c + \tfrac{216}{5}c^2 - 12c^3 + t(72 - \tfrac{864}{5}c + 144c^2 - \tfrac{288}{7}c^3) \\ & - t^2(\tfrac{288}{5} - 144c + \tfrac{864}{7}c^2 - 36c^3) \\ & + t_1(96 - 240c + \tfrac{1440}{7}c^2 - 60c^3) + tt_1(160 - \tfrac{2880}{7}c + 360c^2 - \tfrac{320}{3}c^3) \\ & + t_1^2(\tfrac{800}{7} - 300c + \tfrac{800}{3}c^2 - 80c^3), \end{aligned} \tag{27}$$

$$m_{11} = \tfrac{2}{5} - \frac{c}{3} + t(\tfrac{2}{3} - \tfrac{4}{7}c) + t^2\left(\tfrac{2}{7} - \frac{c}{4}\right) + t_1\left(\tfrac{4}{7} - \frac{c}{2}\right) + tt_1(\tfrac{1}{2} - \tfrac{4}{9}c) + t_1^2\left(\tfrac{2}{9} - \frac{c}{5}\right), \tag{28}$$

$$m_{12} = \tfrac{1}{3} - \tfrac{2}{7}c + t(\tfrac{4}{7} - \tfrac{1}{2}c) + t^2(\tfrac{1}{4} - \tfrac{2}{9}c) + t_1(\tfrac{1}{2} - \tfrac{4}{9}c) + tt_1(\tfrac{4}{9} - \tfrac{2}{5}c) + t_1^2(\tfrac{1}{5} - \tfrac{2}{11}c), \tag{29}$$

$$m_{22} = \tfrac{2}{7} - \tfrac{1}{4}c + t(\tfrac{1}{2} - \tfrac{4}{9}c) + t^2(\tfrac{2}{9} - \tfrac{1}{5}c) + t_1(\tfrac{4}{9} - \tfrac{2}{5}c) + tt_1(\tfrac{2}{5} - \tfrac{4}{11}c) + t_1^2(\tfrac{2}{11} - \tfrac{1}{6}c). \tag{30}$$

The two eigenvalues of the resulting problem,

$$(\mathbf{k} - \omega^2\mathbf{m}) = \mathbf{0}, \tag{31}$$

are functions of the two unknown multipliers and can be optimized with respect to them, by imposing

$$\partial\omega_i^2/\partial t = \partial\omega_i^2/\partial t_1 = 0, \qquad i = 1, 2. \tag{32}$$

In Table 2 the first two frequencies are given, together with the corresponding multipliers, for various taper ratios. For the sake of comparisons, the exact frequencies are also given, as deduced by solving the differential equations of motion [21]. The almost perfect coincidence of the first frequency and the good agreement of the second frequency should

TABLE 2

First two non-dimensional frequencies for cantilever beam with linearly varying height: two-parameter approximation and exact results

c	ω	$\bar{\omega}$	t	t_1
0	3·516	3·516	−0·7682	0·1971
0·1	3·559	3·560	−0·2515	0·0064
0·2	3·608	3·610	−0·0869	−0·0476
0·3	3·667	3·668	0·0856	−0·1081
0·4	3·737	3·738	0·2821	−0·1776
0·5	3·824	3·824	0·3241	−0·2871
0·6	3·934	3·935	−0·1109	−0·1705
0·7	4·082	4·083	−0·0243	−0·2128
0·8	4·293	4·296	0·0823	−0·2313
0·9	4·631	4·634	0·1719	−0·1304
0	22·034	22·158	−0·8649	0·1567
0·1	21·338	21·472	−0·7181	0·0478
0·2	20·621	20·752	−0·5230	−0·0965
0·3	19·881	19·991	−0·2595	−0·2905
0·4	19·114	19·191	0·1008	−0·5534
0·5	18·317	18·357	0·5969	−0·9092
0·6	17·488	17·509	1·2712	−1·3749
0·7	16·625	16·674	2·1246	−1·9052
0·8	15·743	15·872	2·9053	−2·1579
0·9	14·931	15·069	2·5406	−0·8905

be noticed, for all the taper ratios. It is also worth noting that the case $c = 1$ (wedge beam) cannot be solved in terms of Bessel functions.

As a final example in vibration problems, consider a beam of uniform cross-section with flexible ends (Figure 2). A similar structural system was examined by Laura and co-workers [22], but their analysis was restricted to symmetric beams. The governing boundary conditions are

$$v(z=0)=0, \qquad v'(z=0)=c_1'v''(z=0), \tag{33}$$

$$v(z=l)=0, \qquad v'(z=l)=-c_2'v''(z=l), \tag{34}$$

where $c_i' = EIc_i$ and c_i are the flexibility constants at the ends. A suitable approximation function can be

$$f(z)=z(d_0+d_1z+d_2z^2+z^3), \tag{35}$$

where

$$d_1=\frac{l^3+6c_2'l^2}{4(c_1'+c_2')+2l+12c_1'c_2'/l}\equiv\frac{a}{b}, \qquad d_2=-\frac{2c_1'+l}{l^2}\frac{a}{b}-l, \qquad d_0=2c_1'a/b, \tag{36}$$

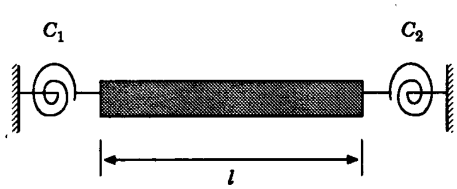

Figure 2. Beam with rotationally flexible ends.

TABLE 3

Non-dimensional frequency for symmetric beam with rotationally flexible ends

$c_1' = c_2'$	k	$\bar{\omega}$	Exact [22]	Laura [22]
0	1·43338	22·3776	22·3732	22·3798
0·1	0·66657	17·2699	17·2693	—
0·2	0·40281	15·1900	15·1894	15·1915
0·4	0·22950	13·3065	13·3054	13·3068
0·5	0·19339	12·7949	12·7937	—
0·6	0·16954	12·4173	12·4160	12·4171
0·8	0·14032	11·8962	11·8950	—
1	0·12328	11·5532	11·5518	11·5527
10	0·06702	10·0670	10·0657	10·0663
100	0·06200	9·8907	9·8895	9·8900
1000	0·06150	9·8728	9·8716	—

and the Rayleigh quotient

$$\bar{\omega}(k) = \left(\int_0^1 v''^2(\zeta)\,\mathrm{d}\zeta + v'^2(0)/c_1 + v'^2(l)/c_2\right)\Bigg/\int_0^1 v(\zeta)^2\,\mathrm{d}\zeta \tag{37}$$

can be conveniently calculated by using the trial function

$$v(z) = f(z)(1 + kf(z)). \tag{38}$$

In Table 3 the first non-dimensional frequency is given for a symmetric ($c_1' = c_2'$) beam with constant cross section and for various values of the flexibility. A comparison is made with the exact values and with the results obtained by Laura *et al.* by taking into account the symmetry of the system. As can be immediately seen, our results are more accurate for low values of the flexibility, so confirming the statement: "If one takes into account the symmetry of the system the calculated eigenvalues are, *in general*, more accurate than those which would result if the complete structural system were analyzed" [22].

In Table 4 the approximate frequencies for two asymmetric beams with constant cross section are compared with the exact values calculated by means of the method reported in reference [23]. In the first case the left end is supposed to be clamped ($c_1' = 0$), whereas

TABLE 4

Non-dimensional frequency for asymmetric beam with rotationally flexible ends

	$c_1' = 0$			$c_1' = 1000$		
c	k	$\bar{\omega}$	Exact [23]	k	$\bar{\omega}$	Exact [23]
0·1	0·9635	19·6302	19·6273	0·2065	13·4368	13·4306
0·2	0·7378	18·4341	18·4292	0·1611	12·4946	12·4910
0·4	0·5508	17·3440	17·3361	0·1210	11·5893	11·5871
0·6	0·4734	16·8345	16·8253	0·1035	11·1493	11·1475
0·8	0·4319	16·5394	16·5295	0·0938	10·8889	10·8874
1	0·4062	16·3468	16·3360	0·0877	10·7168	10·7154
5	0·3206	15·6354	15·6234	0·0669	10·0647	10·0635
10	0·3097	15·5345	15·5223	0·0642	9·97000	9·9688
100	0·2998	15·4410	15·4288	0·0617	9·88179	9·88058
1000	0·2988	15·4316	15·4190	0·0615	9·87280	9·8716

in the other case a large flexibility value ($c_1' = 1000$) simulates a simply supported end. The agreement is quite satisfactory, for all the values of the flexibilities.

3. BUCKLING PROBLEMS

Another interesting application of the optimized Rayleigh quotient is to the solution of buckling problems. Consider first the classical example of a cantilever beam with constant cross section, subjected to a concentrated axial force F at the free end. It is well-known that the exact non-dimensional critical load

$$\mu = FL^2/EI \tag{39}$$

is given by $\pi^2/4$, and that the same problem was solved by means of the Rayleigh approach with one multiplier [24]. A second order approximation can be deduced by using the trial function

$$v(\zeta) = \zeta^2 + k\zeta^4 + k_1\zeta^6, \tag{40}$$

and by running the following *Mathematica* lines:

```
v = z^2 + k z^4 + k1 z^6;
nray = Simplify[Integrate[D[v, {z, 2}]^2,{z, 0, 1}];
dray = Simplify[Integrate[D[v, z]^2,{z, 0, 1}];
ray = Simplify[nray/dray];
Simplify[Solve[{D[ray,k] = = 0,  D[ray, k1] = = 0},{k, k1}]]
ray/.%
```

The Rayleigh quotient

$$\bar{\mu}(k, k_1) = \int_0^1 v''^2(\zeta)\,d\zeta \Big/ \int_0^1 v'^2(\zeta)\,d\zeta \tag{41}$$

will be given by

$$\bar{\mu} = \frac{4 + 16k + (144/5)k^2 + 24k_1 + (720/7)kk_1 + 100k_1^2}{(4/3) + (16/5)k + (16/7)k^2 + (24/7)k_1 + (16/3)kk_1 + (36/11)k_1^2}, \tag{42}$$

and its minimum values are obtained corresponding to the following three pairs of parameters:

$$\begin{aligned}(k, k_1)_1 &= (-2{\cdot}75618246, 1{\cdot}806102419),\\ (k, k_1)_2 &= (-0{\cdot}2042053916, 0{\cdot}01510151945),\\ (k, k_1)_3 &= (-1{\cdot}437122235, 0{\cdot}5364280531).\end{aligned} \tag{43}$$

For the second set of parameters a critical load $\bar{\mu} = 2{\cdot}467401752$ is obtained, which is 0·0136% higher than the true result.

A dramatic improvement can be obtained if the optimized Timoshenko quotient

$$\bar{\mu} = \int_0^1 v'^2(\zeta)\,d\zeta \Big/ \int_0^1 (v(1) - v(\zeta))^2\,d\zeta \tag{44}$$

is used. In fact, as reported in reference [24], a Timoshenko approach with one multiplier gives a critical load which is just 0·026% higher than the exact value, whereas the corresponding Rayleigh quotient is 0·75% higher than the true result.

TABLE 5

Non-dimensional critical load for asymmetric beam with rotationally flexible ends

	$c_1' = c_2'$		$c_1' = 0$		$c_1' = 1000000$	
c_2'	k	$\bar{\mu}$	k	$\bar{\mu}$	k	$\bar{\mu}$
0	5·969958	39·60882	5·969958	39·60882	0·611426	20·808843
0·1	2·083442	28·17446	3·433443	33·299792	0·374269	17·374920
0·2	0·974847	22·679436	2·279776	29·881223	0·257528	15·419430
0·4	0·419815	17·756836	1·467229	26·588241	0·166395	13·473009
0·6	0·267933	15·544423	1·174152	25·030602	0·131149	12·527794
0·8	0·202656	14·297014	1·027913	24·130373	0·113072	11·973491
1	0·167472	13·497999	0·9411831	23·545357	0·102211	11·609942
5	0·077229	10·656556	0·674333	21·412189	0·068323	10·260652
10	0·068476	10·268199	0·642641	21·114321	0·064278	10·069144
100	0·061090	9·911911	0·614525	20·839737	0·060692	9·891909
1000	0·0603768	9·875925	0·611735	20·811933	0·060337	9·873925

Another improvement can be obtained if a two-term Timoshenko approach is used:

```
v = z^2 + k z^4 + k1 z^6;
v1 = v/.z − >1
nray = Simplify[Integrate[D[v, z]^2,{z, 0, 1}];
dray = Simplify[Integrate[(v1 − v)^2,{z, 0, 1}];
ray = Simplify[nray/dray];
Simplify[Solve[{D[ray, k] = = 0,   D[ray, k1] = = 0},{k, k1}]]
ray/.%
```

The quotient can be calculated as

$$\bar{\mu}(k, k_1) = \frac{39(385 + 924k + 660k^2 + 990k_1 + 1540kk_1 + 945k_1^2)}{2(3003 + 6864k + 4004k^2 + 7150k_1 + 8424kk_1 + 4455k_1^2)}, \tag{45}$$

and its minimum values correspond to the following parameter values:

$$(k, k_1)_1 = (-2·880577025, 1·925817278),$$
$$(k, k_1)_2 = (-0·2046221973, 0·01537081063),$$
$$(k, k_1)_3 = (-1·505631194, 0·602364384). \tag{46}$$

The second set of parameters gives the non-dimensional critical load as $\bar{\mu} = 2·467401108746602$, which is just $3·41 \times 10^{-7}$ higher than the true result.

As another example, the critical load for the beam in Figure 2 is calculated:

$$\bar{\mu} = \left(\int_0^1 v''^2(\zeta)\,d\zeta + v'^2(0)/c_1 + v'^2(l)/c_2\right)\bigg/\int_0^1 v'(\zeta)^2\,d\zeta. \tag{47}$$

If the trial function (28) is used, a very long expression is obtained, which is not possible to give here. On the other hand, some numerical results are given in Table 5. The limiting cases reproduce the clamped beam ($c_1' = c_2' = 0$), the clamped-hinged beam ($c_1' = 0$, $c_2' = \infty$) and the simply supported beam ($c_1' = \infty$, $c_2' = \infty$).

Finally, the buckling loads for cantilever and simply supported beams with variable cross-section, in the presence of variable axial forces are calculated. More precisely, the following variation law of the moment of inertia will be considered,

$$I(\zeta) = I_0(1 + \zeta)^p, \tag{48}$$

and the distributed load along the column will be given by

$$q(\zeta) = q_0\zeta^r. \tag{49}$$

Firstly, it is interesting to consider in some detail the concentrated load case, which is defined by the Rayleigh quotient

$$\bar{\mu} = \frac{FL^2}{EI_0} = \int_0^1 (1 + \zeta)^p v''^2(\zeta)\,\mathrm{d}\zeta \bigg/ \int_0^1 v'^2(\zeta)\,\mathrm{d}\zeta, \tag{50}$$

or by the Timoshenko quotient

$$\bar{\mu} = \frac{FL^2}{EI_0} = \int_0^1 (1 + \zeta)^p v'^2(\zeta)\,\mathrm{d}\zeta \bigg/ \int_0^1 (v(1) - v(\zeta))^2\,\mathrm{d}\zeta. \tag{51}$$

Of course, $v(1) = 0$ will result for the simply supported case. The one-parameter results have been obtained by using the trial function

$$v(\zeta) = \zeta^2 + k\zeta^3 \tag{52}$$

for the cantilever beam, and

$$w(\zeta) = \zeta(1 - \zeta), \qquad v(\zeta) = w(\zeta)(1 + kw(\zeta)) \tag{53}$$

for the simply supported beam. A second order approximation can be deduced by employing:

$$v(\zeta) = \zeta^2 + k\zeta^3 + k_1\zeta^4 \tag{54}$$

and

$$w(\zeta) = \zeta(1 - \zeta), \qquad v(\zeta) = w(\zeta)(1 + kw(\zeta) + k_1 w(\zeta)^2). \tag{55}$$

for the cantilever and the simply supported beam, respectively. The results are given in Table 6, together with the exact values, as given in reference [25] for $p = 1$ and $p = 2$. As can be easily seen, the Timoshenko quotient behaves better than the

TABLE 6

Non-dimensional critical load for cantilever beam and simply supported beam with varying cross-section, subjected to concentrated load at the end

Beam	Exact [25]	RS1	RS2	TS1	TS2
		$p = 1$			
Free-clamped	3·1176962	3·12053	3·117928	3·117754	3·1176998
Clamped-free	4·1241844	4·21553	4·125455	4·127228	4·12421
Hinged-hinged	14·51125	14·8126	14·8044	14·5843	14·58426
		$p = 2$			
Free-clamped	3·8363769	3·92963	3·83785	3·83891	3·836394
Clamped-free	6·7318654	6·96578	6·75393	6·73989	6·732289
Hinged-hinged	20·792288	22·5518	22·5221	21·21665	21·21653
		$p = 3$			
Free-clamped	—	5·01494	4·6347	4·62251	4·612390
Clamped-free	—	11·0944	10·80128	10·70591	10·6938
Hinged-hinged	—	34·8021	34·6743	30·3637	30·36157

TABLE 7

Non-dimensional critical load for cantilever beam with varying cross-section, subjected to distributed axial load

r \ p	0	1	2	3	4	5
0	7·83797 (7·83735)	16·103115 (16·10095)	27·2607 (27·25691)	41·3101 (41·30481)	58·2512 (58·24450)	78·0844 (78·07591)
1	13·8872 (13·88629)	29·4061 —	50·6998 —	77·7720 —	110·6243 —	149·2581 —
2	24·2878	53·2774	93·8017	145·8796	209·5159	284·7132
3	41·8354	95·6480	172·516	272·503	395·615	541·853

Rayleigh quotient, as already stated in reference [26]. Consequently, a more detailed analysis of the free-clamped beam was performed by using a two-parameter Timoshenko quotient. Similar analyses for other boundary conditions pose no difficulties. In Table 7 the non-dimensional critical load

$$\mu = \frac{L^2}{EI_0}\int_0^l q_0\zeta^r \, d\zeta \tag{56}$$

is given for various values of the parameters r and p. Comparisons can only be made for the cases $r = 0$ and $r = 1$, $p = 0$, and show good agreement between exact [25] and approximate results.

4. FREQUENCY-AXIAL LOAD CURVES

Consider now a vibrating cantilever beam in the presence of a concentrated axial load F at the tip. If the force is assumed to be conservative, then the first vibration frequency can be given by the Rayleigh quotient [27]: e.g.,

$$\bar{\omega}^2 = \left(\int_0^1 EIv''^2(\zeta)\, d\zeta - F\int_0^1 v'^2(\zeta)\, d\zeta\right)\Big/\int_0^1 \rho A v^2(\zeta)\, d\zeta. \tag{57}$$

If the height of the cross section is assumed to vary according to a linear law (see Figure 3), then the area of the moment of inertia of the cross section will be given by

$$A(\zeta) = A_0(1 - c\zeta), \qquad I(\zeta) = I_0(1 - c\zeta)^3, \tag{58}$$

respectively, where c is a parameter which can also be negative. In order to obtain a first-order approximation of the frequency-load relationship, a one-parameter optimized Rayleigh method is used with trial function

$$v(\zeta) = \zeta^2 + k\zeta^3, \tag{59}$$

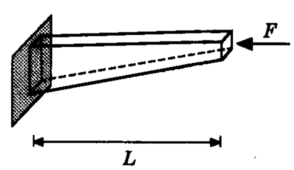

Figure 3. Cantilever beam with linearly varying height, subjected to axial force at the tip.

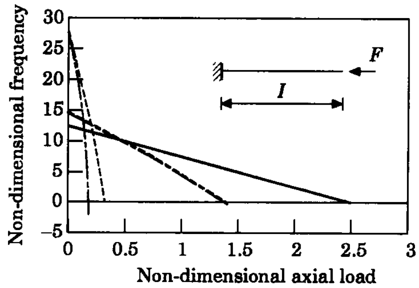

Figure 4. Axial load-frequency relationship for a cantilever beam with linearly varying breadth. —, RS1, $c = 0$; - - - -, RS1, $c = 0{\cdot}5$ (smaller slope curve); – – – –, $c = 1{\cdot}0$ (larger slope curve); — — —, RS2, $c = 0$; – ·· – ·· –, RS2, $c = 0{\cdot}5$; – · – · –, RS2, $c = 1{\cdot}0$.

to obtain the quotient

$$\bar{\omega}^2(k) = \frac{EI_0}{\rho A_0} \frac{A + Bk + Ck^2}{168 - 140c + (280 - 240c)k + (120 - 105c)k^2}, \tag{60}$$

where

$$A = 3360 - 5040c + 3360c^2 - 840c^3 - 1120\alpha^2, \tag{61}$$

$$B = 10080 - 20160c + 15120c^2 - 4032c^3 - 2520\alpha^2, \tag{62}$$

$$C = 10080 - 22680c + 18144c^2 - 5040c^3 - 1512\alpha^2, \tag{63}$$

$$\alpha^2 = F/EI_0. \tag{64}$$

The usual procedure leads to the two parameter values

$$k_{1,2} = (R \pm 6\sqrt{S^2 - 20TU})/90T \tag{65}$$

where

$$R = EI_0(-46080 + 152280c - 188964c^2 + 104760c^3 - 22050c^4) + F(4272 - 3360c), \tag{66}$$

$$S = EI_0(7680 - 25380c + 31494c^2 - 17460c^3 + 3675c^4) - F(712 - 560c), \tag{67}$$

$$T = EI_0(640 - 2100c + 2616c^2 - 1466c^3 + 312c^4) - F(48 - 39c), \tag{68}$$

$$U = EI_0(3360 - 11520c + 14340c^2 - 7824c^3 + 1620c^4) - F(490 - 375c). \tag{69}$$

In Figure 4 some frequency-axial load curves are sketched, for various values of the c parameter. In the same figure, the results of a two-parameter Rayleigh approach are reported, and the differences seem to be noticeable only for the conical beam ($c = 1$).

Finally, in Figure 5 the same curves as above are reported, for a beam with linearly varying breadth. In this case, the two-parameter refinement seems to be unnecessary.

5. CONCLUSIONS

The aim of this paper was twofold: first of all, to show all the potentialities of the Rayleigh and Timoshenko quotients in their non-integer multiplier version, and then to obtain—by using the *Mathematica* symbolic language—close approximations to some

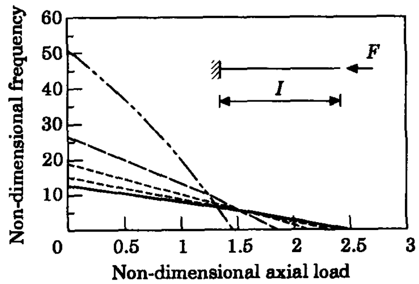

Figure 5. Axial load-frequency relationship for a cantilever beam with linearly varying height. ——, $c = 0$; - - - -, $c = 0·25$ (lower slope curve); – – – –, $c = 0·5$ (larger slope curve); — — —, $c = 0·75$; – ·· – ·· –, $c = 1·0$.

classical problems in vibrations and buckling analysis. The following conclusions can be drawn.

(1) The Timoshenko approach is noticeably more powerful than the corresponding Rayleigh method, at least for the problems considered in this paper.

(2) The rate of convergence of the quotients is generally so fast that the two-parameter version gives very satisfactory results.

(3) The use of the *Mathematica* software allows one to calculate easily quotients with unknown parameters, as for example buckling loads of beams with non-constant cross-section or with elastic ends.

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