Journal of Sound and Vibration (1990) **137**(1), 107–115

THE INFLUENCE OF AN INTERMEDIATE SUPPORT ON THE STABILITY BEHAVIOUR OF CANTILEVER BEAMS SUBJECTED TO FOLLOWER FORCES

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(*Received* 13 *July* 1988, *and in revised form* 13 *June* 1989)

A non-uniform undamped beam, which is simply supported at one end, resting on an elastically flexible support at an intermediate abscissa q, is examined. The beam is subjected to a generic set of follower forces along the axis, so that Beck, Leipholz and Hauger problems can be easily recovered as particular cases. In all these systems a location for the intermediate support exists, which corresponds to passage from flutter to divergence.

1. INTRODUCTION

Great attention was recently devoted to non-conservative systems in which instability occurs by divergence in one range of a parameter and by flutter in another range. The Beck rod with elastic restraints was extensively investigated by Kounadis *et al.* [4–8], who sketched the parametric regions corresponding to divergence and to flutter. The more complex system shown in Figure 2 was first examined by Zorii and Chernukha [1]. According to their investigations, dynamic instability occurs if the support is placed at $q < 0{\cdot}418l$, and static instability otherwise. Quite recently, Elishakoff and Hollkamp [2] studied the same system by means of an approximate Galerkin-type procedure, resulting in a transition value q between flutter and divergence equal to $0{\cdot}38647l$. In both these papers a marked discontinuity of the critical load at the transition value was found. Finally, Elishakoff and Lottati [3] were able to find the exact solution of the problem, confirming the transition value $q = 0{\cdot}418l$. On the other hand, their work indicated that no discontinuity exists at the transition value, even if the flutter boundary seems to have a vertical slope.

In the present paper a general discretization method of analysis is proposed, which allows us to study the system in Figure 1 with arbitrarily varying cross-section, and subjected to arbitrarily distributed follower forces along the axis. A number of particular cases can be immediately recovered; as, for example, the generalized Beck rod of Figure 1 and the generalized Leipholz rod of Figure 2. The transition value for the generalized Beck rod was found to be equal to $0{\cdot}417l$, thus confirming the value given by Zorii and Chernukha and by Elishakoff and Lottati. The passage from divergence to flutter is associated with a noticeable lowering of the critical load.

2. THE EQUATION OF MOTION

Consider the rod shown in Figure 3, the span of which is equal to l, the cross-section of which can vary with a generic—even discontinuous—law. The constraint at A has a

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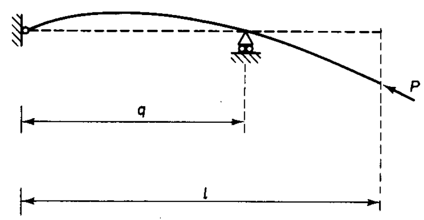

Figure 1. Generalized Beck's rod.

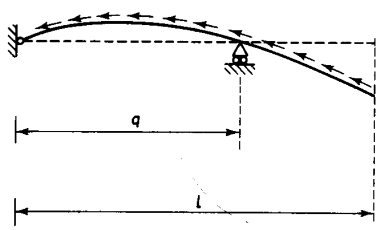

Figure 2. Generalized Leipholz's rod.

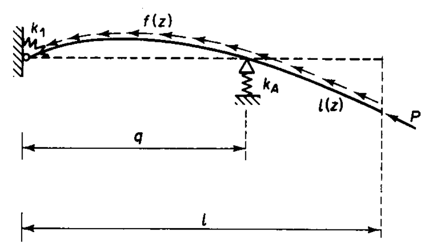

Figure 3. Generalized combined Beck-Leipholz problem with elastically flexible support.

rotational stiffness k_1, while the support at B has an extensional stiffness k_B. Let $q' = q/l$ be the non-dimensional abscissa of this intermediate support.

The applied loads consist of a concentrated force P at the top, and of a generally varying distributed follower forces $p(z)$ along the span. The structure will be discretized by means of the so-called "cells method", which was introduced by Raithel and Franciosi [9, 10], and was subsequently used by Franciosi *et al.* to investigate the static and dynamic behaviour of suspension bridges, [11-13], monumental masonry arches [14], and beams on Winkler soil and Green soil [15-17].

The system will be reduced to a finite number n of rigid bars, connected together by means of $n+1$ elastic "cells", in which the strain energy of the structure is supposed to be concentrated. In this way a holonomic n-degree-of-freedom system is obtained, the equation of motion of which can be deduced from the Lagrange equations:

$$\mathbf{M}\ddot{\mathbf{c}} + (\mathbf{K} - \lambda_1 \mathbf{P}_1 - \lambda_2 \mathbf{P}_2)\mathbf{c} = \mathbf{0}. \tag{1}$$

The symbols appearing in equation (1) have the following meanings: $\mathbf{M}$ is the Lagrangian mass matrix, $\mathbf{K}$ is the global stiffness matrix of the system, $\mathbf{c}$ is the vector of the n Lagrangian co-ordinates, λ_1 and λ_2 are the load multipliers of the concentrated load at the top and of the distributed loads, respectively, $\mathbf{P}_1$ and $\mathbf{P}_2$ are second order instability matrices associated with the concentrated force and to the distributed loads respectively.

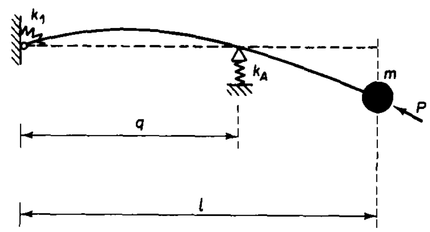

Figure 4. Pflüger system.

The matrices **M** and **K** are symmetric and positive definite, whereas the matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ are not symmetric. Consequently the problem (1) is not self-adjoint, and the resulting eigenvalue equation,

$$-\omega^2\mathbf{M}+(\mathbf{K}-\lambda_1\mathbf{P}_1-\lambda_2\mathbf{P}_2)x=0, \tag{2}$$

could exhibit complex eigenvalues ω_i^2.

In the following the rotations ϕ_i of the n rigid bars will be taken as the Lagrangian co-ordinates. The kinetic energy, the strain energy and the virtual work of the non-conservative forces are calculated as follows.

3. THE KINETIC ENERGY

If the masses are supposed to be lumped at the cells abscissae, the resulting $n+1$ concentrated masses can be ordered into the diagonal matrix $\mathbf{M}'=\text{diag}\{m_i\}$. The kinetic energy is therefore:

$$T=\tfrac{1}{2}\dot{\mathbf{v}}^{\mathrm{T}}\mathbf{M}'\dot{\mathbf{v}}, \tag{3}$$

where $\mathbf{v}$ is the $(n+1)$-dimensional vector of the displacements of the cells abscissae, and the dot denotes differentiation with respect to time. These displacements can be easily expressed as functions of the Lagrangian co-ordinates $\mathbf{c}$, by means of the rectangular transfer matrix $\mathbf{V}$:

$$\mathbf{v}=\mathbf{V}\mathbf{c}. \tag{4}$$

If equation (4) is substituted back into equation (3) one obtains the following expression for the kinetic energy:

$$T=\tfrac{1}{2}\dot{\mathbf{c}}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{M}'\mathbf{V}\dot{\mathbf{c}}=\tfrac{1}{2}\dot{\mathbf{c}}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{c}}. \tag{5}$$

The Lagrangian mass matrix **M** is symmetric, but in general it is a full matrix.

Finally, note that the proposed discretization method allows one to study Pflüger system very easily (see Figure 4).

4. THE STRAIN ENERGY

For the sake of simplicity, it is supposed that the rigid bars have equal length l_t and that the local stiffness coefficients of the $n-1$ internal cells are given by (cf. [9])

$$k_i=EI_i/l_t. \tag{6}$$

where E is the Young's modulus of the beam, and I_i is the cross-sectional inertia of the beam at the cell abscissa. The stiffness of the last cell is equal to zero, while the stiffness of the first cell is given by

$$k_1=2EI_1/(2c_1EI_1+l_t), \tag{7}$$

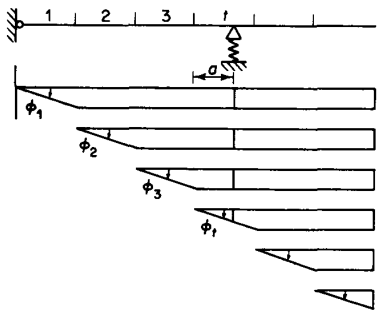

Figure 5. Displacement of the elastically flexible support as a function of the Lagrangian co-ordinates.

where c_1 is the rotational flexibility of the constraint. If $c_1 = 0$ the constraint is a clamped end, whereas $c_1 = \infty$ defines a hinged end. The strain energy of the system is sum of the bending strain energy of the cells, and of the axial strain energy of the flexible support:

$$L = \tfrac{1}{2}\mathbf{c}^T\mathbf{K}_b\mathbf{c} + \tfrac{1}{2}\mathbf{c}^T\mathbf{K}_e\mathbf{c}. \tag{8}$$

The matrix $\mathbf{K}_b$ is a typical tridiagonal matrix, the terms of which are given by

$$K_{ii} = k_i + k_{i+1}, \qquad K_{i+1,i} = K_{i,i+1} = -k_{i+1}. \tag{9}$$

The axial strain energy L_e is given by

$$L_e = \tfrac{1}{2}k_A v_A^2, \tag{10}$$

where k_A is the stiffness of the intermediate support, and v_A is its displacement. This displacement can be expressed, as a function of the Lagrangian co-ordinates, as follows (see Figure 5):

$$v_A = l_t \sum_{i=1}^{t-1} \phi_i + a\phi_t. \tag{11}$$

Here t is the index of the rigid bar which contains the support.

5. THE VIRTUAL WORK OF THE NON-CONSERVATIVE FORCES

First of all, the distributed forces along the axis must be reduced to concentrated forces at the cells abscissae. Thus one can consider the deformed system in Figure 6, which is

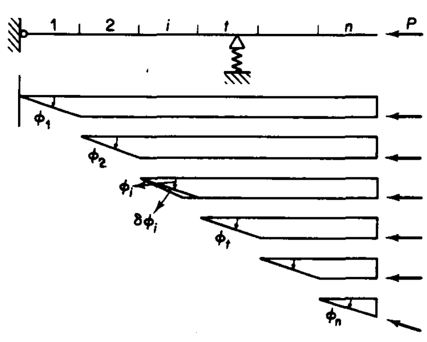

Figure 6. Virtual work of the follower forces.

completely defined by the vector **c** of the Lagrangian co-ordinates, and assign a virtual variation $\delta\phi_i$ to the ith Lagrangian co-ordinate. The virtual work of the unitary concentrated force at the top is

$$\delta W' = (\delta\phi_i - \delta\phi_n)l_t, \tag{12}$$

and consequently the matrix $\mathbf{P}_1$ will be

$$\mathbf{P}_1 = \begin{bmatrix} l_t & 0 & 0 & \dots & -l_t \\ 0 & l_t & 0 & \dots & -l_t \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & l_t & -l_t \\ 0 & 0 & \dots & \dots & 0 \end{bmatrix}. \tag{13}$$

In the same way, a follower force at the cell j will be associated to the matrix

$$\mathbf{P}_{2j} = \begin{bmatrix} l_t & 0 & 0 & -l_t & 0 & \dots & 0 \\ 0 & l_t & 0 & -l_t & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & l_t & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ j-1 \\ \\ \end{matrix}. \quad (\text{column } j-1) \tag{14}$$

Finally, the matrix $\mathbf{P}_2$ is given by the sum

$$\mathbf{P}_2 = \sum_{j=1}^{n-1} F_j P_{2j}. \tag{15}$$

6. NUMERICAL EXAMPLES

Consider the structure shown in Figure 1, which has already been extensively studied in references [1] and [2]. All the numerical results in this section have been obtained with the beam divided into 20 rigid bars, so that the discretized system has 20 degrees of freedom. It is perhaps worth noting that finer divisions are useless, because they lead to negligible increments in the critical values.

The cross-sectional inertia is assumed to be constant, so that the limiting case $q \to 0$ gives the well known Beck's rod. In this system the critical load is

$$\beta = P_{\text{crit}} l^2 / EI = 20{\cdot}05, \tag{16}$$

and instability is reached by flutter.

The critical coefficients β are given in Figure 7 for q' varying from 0 to 0·8, and from the graph it is easy to deduce that the values reported in reference [2] are rather over-estimated. Moreover, the transition value is given by $q' = 0{\cdot}4165$, so confirming the value $q' = 0{\cdot}418$ given in references [1] and [3]. In Figures 8 and 9 the results for the first two frequencies *vs.* the axial load multiplier are shown, for $q' = 0{\cdot}416$ (flutter) and $q' = 0{\cdot}417$ (divergence), respectively. From these figures it is also possible to explain the sudden increase in the critical load value as q' increases past 0·4165.

As a second example, consider the generalized Leipholz rod in Figure 2. In Figure 10 the critical β value is given, for different q values. In this system, $q' \to 0$ is the limiting

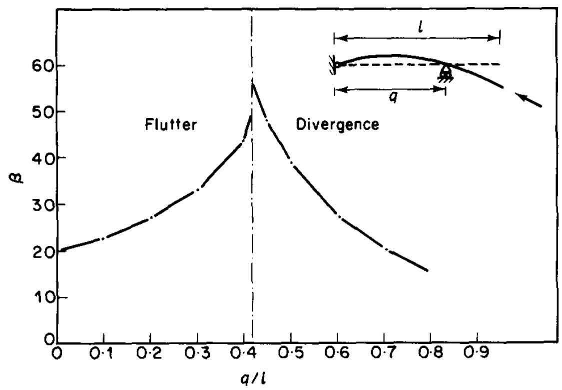

Figure 7. Flutter and divergence regions for the generalized Beck's rod as a function of the intermediate support abscissa.

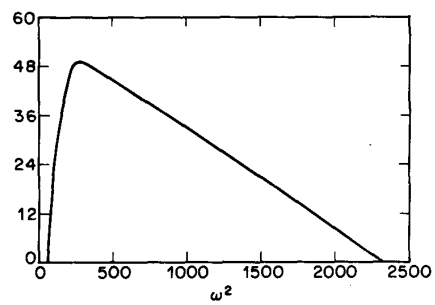

Figure 8. Flutter load of the generalized Beck's rod at $q' = 0{\cdot}416$ (flutter load $= 48{\cdot}97$).

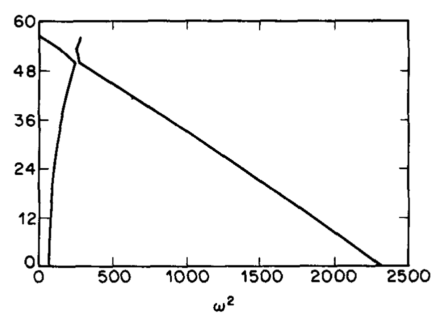

Figure 9. Divergence load of the generalized Beck's rod at $q' = 0{\cdot}417$ (divergence load $= 56{\cdot}104$).

case of the Leipholz cantilever beam and [18, see pp. 329-332]

$$\beta_{fl} = p_{\text{crit}} l^3 / EI = 40{\cdot}05, \tag{17}$$

while the other limiting case $q' \to 1$ furnishes the simply supported Leipholz rod. In this latter case it is [18, see pp. 329-332]

$$\beta_{\text{div}} = 18{\cdot}96. \tag{18}$$

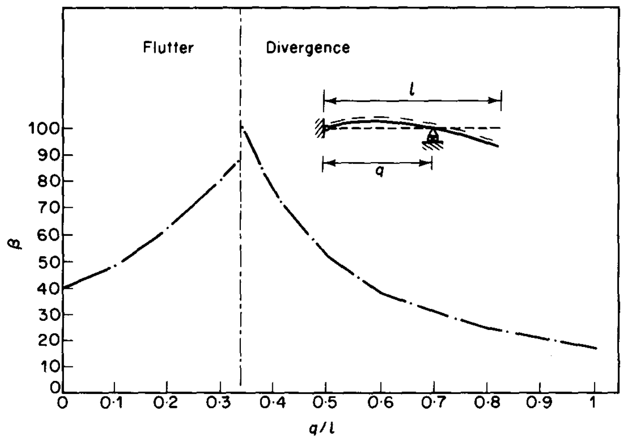

Figure 10. Flutter and divergence regions for the generalized Leipholz's rod as a function of the intermediate support abscissa.

The qualitative behaviour is quite similar to the behaviour of the generalized Beck's rod, and the double critical point is here attained for $q' = 0{\cdot}335$.

As a final example, consider Hauger's rod [18, see pp. 337–343] which is subjected to a linearly distributed load over the length of the rod:

$$g = g_0(l - x). \tag{19}$$

In order to show the convergence properties of the proposed method of analysis, one can consider some classical cases, in which no intermediate support exists. In Table 1 the non-dimensional coefficient

$$\beta = g_0 l^4 / EI \tag{20}$$

is shown, for various numbers of Lagrangian co-ordinates, and for various boundary conditions. Only the hinged-hinged beam leads to divergence, whereas all the other cases are truly non-conservative systems.

Therefore, consider the structure in Figure 11, where the intermediate support is placed at the generic abscissa. In Figure 12 the critical value of the load multiplier is plotted against the value of the abscissa of the support; even in this case numerical results suggest that there is a marked discontinuity between flutter and divergence, at $q = 1{\cdot}685l$.

TABLE 1

Non-dimensional critical coefficients of the classical Hauger's rod as a function of the Lagrangian co-ordinates

n	10	20	30	50	Leipholz
Clamped-free	142·6	148·83	149·77	150·2	150·8
Hinged-hinged	61·13	61·67	61·80	61·85	61·87
Clamped-clamped	328·3	363·0	369·7	373·1	375·2
Clamped-hinged	283·6	306·16	310·16	312·22	313·6

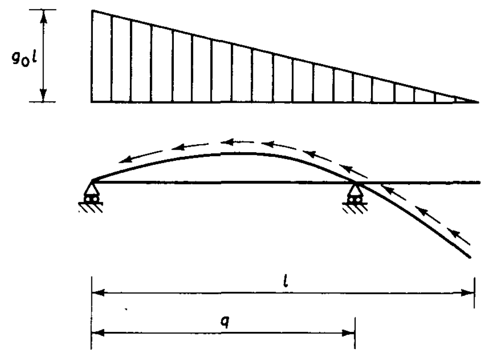

Figure 11. Generalized Hauger's rod.

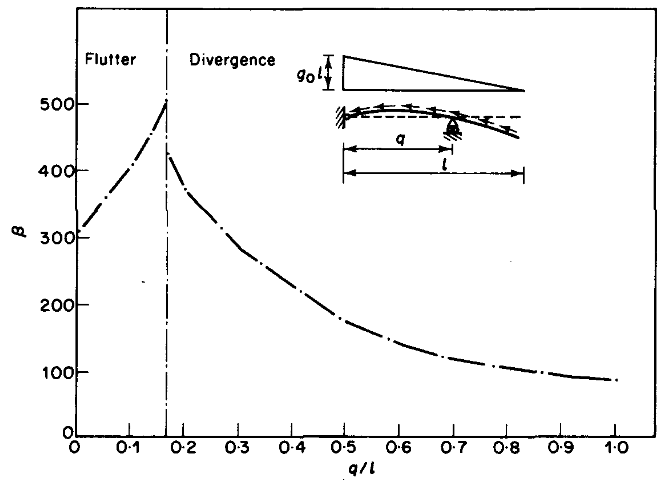

Figure 12. Flutter and divergence regions for the generalized Hauger's rod as a function of the intermediate support abscissa.

7. CONCLUSIONS

A simple method has been proposed, which allows one to calculate the critical (divergence or flutter) load multiplier for a beam with a hinge at one end and with an elastically flexible support at some intermediate abscissa. According to this method, non-uniform beams can be studied, subjected to an arbitrary set of follower forces along the axis.

Some numerical examples have been presented; in the first one an already examined Beck's system is revisited, while in the second one a generalized Leipholz's structure is examined. Finally, a generalized version of the Hauger rod is introduced, which allows one to show some convergence properties of the proposed method.

ACKNOWLEDGMENT

The authors wish to express their gratitude to the referee who brought reference [3] to their attention and provided many suggestions for improving this paper.

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