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# Some finite elements for the static analysis of beams with varying cross section

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**Abstract**

In this paper three finite elements are proposed, for the static analysis of beams with varying cross section. The shape functions are derived by using a solution proposed in Ref. [1], and the resulting stiffness matrix is generated by means of a symbolic software. The consistent loads are also given, equivalent to a constant distributed load. The performances of the three proposed finite elements are checked for some numerical examples, and the computational gain is clearly shown. © 1998 Elsevier Science Ltd. All rights reserved.

*Keywords:* Finite elements; Euler beams; Varying cross section

## 1. Notation

| | |
|---|---|
| **A** | column vector of the Lagrangian coordinates |
| $b_0$ | width of the element at the left end |
| $b_1$ | width of the element at the right end |
| $b(z)$ | width of the cross section |
| **B** | deformation vector |
| **d** | column vector of the nodal coordinates |
| **f** | column vector of the consistent nodal forces |
| $E$ | Young's modulus |
| $h_0$ | depth of the element at the left end |
| $h_1$ | depth of the element at the right end |
| $h(z)$ | depth of the cross section |
| **k** | element stiffness matrix |
| $I(z)$ | second moment of area |
| $L$ | span of the finite element |
| **N** | row vector of the shape functions |
| $q$ | uniformly distributed load |
| $v(z)$ | vertical displacement |
| $\alpha$ | row vector of the coordinate functions |
| $\beta$ | taper ratio for the width |
| $\eta$ | taper ratio for the depth |
| $\phi(z)$ | rotation of the cross section |

## 2. Introduction

One of the commonest structural members is the beam with rectangular varying cross section, in which the depth and/or the width of the section is assumed to vary according to a simple law. In a finite element context, those non-uniform member is often approximated by a large number of small uniform elements, replacing the continuous variation with a step law.

In this way it is always possible to obtain acceptable results, and the error can be reduced as much as desired by refining the mesh. Nevertheless, the computational effort can become excessive, and it is sometimes more convenient to introduce *ad hoc* finite elements, at least for the more common shape variations.

In this note an elegant solution given in Ref. [1] is used, in order to study the cross sections with linearly varying width, with linearly varying depth and with quadratic variation in depth.

A powerful symbolic software is used [2], and the stiffness matrices are generated, together with the consistent load column matrix, equivalent to a constant distributed load.

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Finally, some numerical examples show the usefulness and precision of these elements, and their convenience from a computational point of view.

## 3. The Shape Functions

Let us consider a two-noded element with two degrees of freedom at each node, so that the nodal displacements will be defined by:

$$\mathbf{d}^T = \{v_1, \phi_1, v_2, \phi_2\} \tag{1}$$

and $\phi = -\mathrm{d}v/\mathrm{d}z$. This choice allows us to immediately use the proposed elements in a more classical context, together with the well-known cubic hermitian finite elements.

The deflection $v$ can be written as:

$$v = \boldsymbol{\alpha}\mathbf{A} \tag{2}$$

where $\mathbf{A}^T = \{C_1, C_2, C_3, C_4\}$ is the column vector of the Lagrangian coordinates, and $\alpha$ is the row column of the coordinate functions, which is given by the following expressions [1]:

*Case A*—cross section with linearly varying width according to the law

$$b(z) = b_0 + \beta z \tag{3}$$

and $\beta$ is the taper ratio,

$$\beta = \frac{b_1 - b_0}{L} \tag{4}$$

where $b_0$ and $b_1$ are the width at the left end and right end, respectively. In this case the coordinate functions are given by:

$$\boldsymbol{\alpha} = [1, b_0 + \beta z, (b_0 + \beta z)\log(b_0 + \beta z), (b_0 + \beta z)^2]. \tag{5}$$

*Case B*—cross section with linearly varying depth according to the law

$$h(z) = h_0 + \eta z \tag{6}$$

and $\eta$ is the taper ratio

$$\eta = \frac{h_1 - h_0}{L} \tag{7}$$

where $h_0$ and $h_1$ are the depth at the left end and right end, respectively.

In this case $\alpha$ can be written as:

$$\boldsymbol{\alpha} = [1, \log(h_0 + \eta z), (h_0 + \eta z), (h_0 + \eta z)^{-1}]. \tag{8}$$

*Case C*—cross section with binomial form parabola variation in depth, according to the law

$$h(z) = (h_0^{1/2} + \eta z)^2 \tag{9}$$

and $\eta$ is now given by

$$\eta = \frac{h_1^{1/2} - h_0^{1/2}}{L}. \tag{10}$$

Therefore, in this last case:

$$\boldsymbol{\alpha} = [1, h_0^{1/2} + \eta z, (h_0^{1/2} + \eta z)^{-3}, (h_0^{1/2} + \eta z)^{-4}]. \tag{11}$$

The shape functions can be obtained by imposing:

$$v(0) = v_1 \quad -v'(0) = \phi_1 \quad v(L) = v_2 \quad -v'(L) = \phi_2. \tag{12}$$

The resulting system $\mathbf{CA} = \mathbf{d}$ can be easily solved, and the shape functions are given by $\mathbf{N} = \alpha\mathbf{C}^{-1}$.

The matrix multiplication will give the following results:

*Case A*

$$\begin{aligned}
N_1(z) =& \frac{1}{DL}[2\beta L(z-L) - \beta(L-z)^2\log(b_0) \\
&+ (2b_0L + \beta L^2 + \beta z^2)\log(b_0 + \beta L) \\
&- 2L(b_0 + \beta z)\log(b_0 + \beta z)] \\
N_2(z) =& \frac{1}{DL}[\beta Lz(L-z) - b_0(L-z)^2\log(b_0) \\
&- z(2b_0L + \beta L^2 - b_0 z)\log(b_0 + \beta L) \\
&+ L^2(b_0 + \beta z)\log(b_0 + \beta z)] \\
N_3(z) =& \frac{1}{DL}[-2\beta Lz + (\beta z^2 - 2b_0L - 2\beta Lz)\log(b_0) \\
&- \beta z^2\log(b_0 + \beta L) \\
&+ 2L(b_0 + \beta z)\log(b_0 + \beta z)] \\
N_4(z) =& \frac{1}{DL}[\beta Lz(z-L) + (z^2(\beta L + b_0) \\
&- L^2(\beta z + b_0))\log(b_0) - z^2(b_0 + \beta L)\log(b_0 + \beta L) \\
&+ L^2(b_0 + \beta z)\log(b_0 + \beta z)]
\end{aligned} \tag{13}$$

and

$$D = (2b_0 + \beta L)[\log(b_0 + \beta L) - \log(b_0)] - 2L. \tag{14}$$

*Case B*

$$\begin{aligned}
N_1(z) =& \frac{1}{D_1(h_0 + \eta z)}\{\eta(2h_0 + \eta z)(z - L) \\
&+ (2h_0 + \eta L)(h_0 + \eta z)[\log(h_0 + \eta L) \\
&- \log(h_0 + \eta z)]\} \\
N_2(z) =& \frac{h_0}{LD_1(h_0 + \eta z)}\{\eta Lz(L-z) - h_0(L-z)^2\log(h_0) \\
&+ z(h_0 z - 2h_0L - \eta L^2)\log(h_0 + \eta L) \\
&+ L^2(h_0 + \eta z)\log(h_0 + \eta z)\} \\
N_3(z) =& \frac{1}{D_1(h_0 + \eta z)}\{-nz(2h_0 + \eta L + \eta z) \\
&+ (2h_0 + \eta L)(h_0 + \eta z)[\log(h_0 + \eta z) \\
&- \log(h_0)]\}
\end{aligned} \tag{15}$$

$$N_4(z) = \frac{h_0 + \eta L}{LD_1(h_0 + \eta z)}\{\eta Lz(z - L) + L^2(h_0 + \eta z)[\log(h_0 + \alpha z) - \log(h_0)] - z^2(h_0 + \eta L)[\log(h_0 + \eta L) - \log(h_0)]\} \quad (16)$$

with

$$D_1 = (2h_0 + \eta L)[\log(h_0 + \eta L) - \log(h_0)] - 2\eta L. \quad (17)$$

*Case C*—In this case it is convenient to define $s = \sqrt{(h_0)}$, in order to simplify the following expressions:

$$N_1(z) = \frac{s^3(z - L)^2}{L^3 D_2(s + \eta z)^4}[4\eta^5 L^3 z(L^2 + 2Lz + 3z^2) + s\eta^4 L^2(L^3 + 34L^2 z + 67Lz^2 + 36z^3) + s^2\eta^3 L(8L^3 + 128L^2 z + 152Lz^2 + 36z^3) + s^3\eta^2(28L^3 + 216L^2 z + 134Lz^2 + 12z^3) + 40s^4\eta(L^2 + 4Lz + z^2) + 20s^5(L + 2z)]$$

$$N_2(z) = \frac{-s^4 z(z - L)^2}{L^2 D_2(s + \eta z)^4}[\eta^4 L^2(L^2 + 2Lz + 3z^2) + 8s\eta^3 L(L + z)^2) + 2s^2\eta^2(14L^2 + 16Lz + 3z^2) + 20s^3\eta(20L + z) + 20s^4]$$

$$N_3(z) = \frac{(s + \eta L)^3 z^2}{L^3 D_2(s + \eta z)^4}[\eta^5 L^4 z^2 + s\eta^4 L^3 z(4L + 5z) + 2s^2\eta^3 L^2(3L^2 + 10Lz + 5z^2) + 2s^3\eta^2(15L^3 + 20L^2 z + 5Lz^2 - 6z^3) + 20s^4\eta(3L^2 + 2Lz - 2z^2) + 20s^5(3L - 2z)]$$

$$N_4(z) = \frac{(s + \eta L)^4 z^2}{L^2 D_2(s + \eta z)^4}[\eta^4 L^2 Z^2(L - z) + 4s\eta^3 Lz(L^2 - z^2) + 2s^2\eta^2(3L^3 + 5L^2 Z^2 - 5Lz^2 - 3z^3) + 20s^3\eta(L^2 - z^2) + 20s^4(Lz)] \quad (18)$$

and

$$D_2 = 20s^4 + 40\eta Ls^3 + 28\eta^2 s^2 L^2 + 8\eta^3 sL^3 + \eta^4 L^4. \quad (19)$$

## 4. The Stiffness Matrix

The deformation vector **B** can be obtained by performing a double differentiation of the shape functions with respect to the spatial coordinate, and the stiffness matrix turns out to be proportional to the outer product of the deformation vector, according to the following formula:

$$\mathbf{k} = E\int_0^L I(z)\mathbf{B}^T\mathbf{B}dz \quad (20)$$

where $E$ is the Young modulus and $I(z)$ is the second moment of area. It is then possible to write down the terms of the stiffness matrix, as follows:

*Case A*

$$k_{11} = \frac{\beta^2 Eh^3}{6DL}[\log(b_0 + \beta L) - \log(b_0)]$$

$$k_{12} = \frac{\beta Eh^3}{6DL}[b_0(\log(b_0 + \beta L) - \log(b_0)) - \beta L]$$

$$k_{14} = -\frac{\beta Eh^3}{6DL}[(b_0 + \beta L)(\log(b_0 + \beta L) - \log(b_0)) - \beta L]$$

$$k_{22} = \frac{Eh^3}{12DL}[\beta L(\beta L - 2b_0) + 2b_0^2(\log(b_0 + \beta L) - \log(b_0))$$

$$k_{23} = \frac{\beta Eh^3}{6DL}[\beta L - b_0(\log(b_0 + \beta L) - \log(b_0))]$$

$$k_{24} = \frac{Eh^3}{12DL}[\beta L(\beta L + 2b_0) - 2b_0(b_0 + \beta L)(\log(b_0 + \beta L) - \log(b_0))]$$

$$k_{44} = \frac{Eh^3}{12DL}[2(b_0 + \beta L)^2(\log(b_0 + \beta L) - \log(b_0)) - \beta L(2b_0 + 3\beta L)]$$

$$k_{13} = -k_{11} \quad k_{33} = k_{11} \quad k_{34} = -k_{14} \quad (21)$$

Table 1
Vertical deflection at the free end of a cantilever beam. First column: cases of cross section variation laws; second column: proposed method with a single element; third column: proposed method with two elements; fourth column: proposed method with five elements; fifth column: classical approach with three cubic hermitian elements; sixth column: classical approach with 10 cubic hermitian elements; seventh column: classical approach with 100 cubic hermitian elements; eighth column: classical approach with 200 cubic hermitian elements

| Case | Nonclassical | | | Classical | | | |
|---|---|---|---|---|---|---|---|
| | $n = 1$ | $n = 2$ | $n = 5$ | $n = 3$ | $n = 10$ | $n = 100$ | $n = 200$ |
| A | 3.15715 | 3.15715 | 3.15715 | 3.28569 | 3.16841 | 3.157260 | 3.157176 |
| B | 1.54308 | 1.54308 | 1.54308 | 1.84350 | 1.56832 | 1.543330 | 1.543145 |
| C | 2.41424 | 2.41422 | 2.41421 | 2.99927 | 2.46085 | 2.414674 | 2.414329 |

*Case B*

$$k_{11} = \frac{Eb\eta^3}{12D_1}(2h_0 + \eta L); \quad k_{12} = -\frac{Eb\eta^3 h_0 L}{12D_1};$$

$$k_{13} = -k_{11} \; k_{14} = -\frac{Eb\eta^3 L}{12D_1}(h_0 + \eta L); \quad k_{23} = -k_{12};$$

$$k_{33} = k_{11}; \quad k_{34} = -k_{14} \; k_{22} = \frac{Ebh_0^2}{12LD_1}[2(h_0 + \eta L)^2$$
$$\times (\log(h_0 + \eta L) - \log(h_0)) - 2\eta h_0 L - 3\eta^2 L^2]$$

$$k_{24} = \frac{Ebh_0(h_0 + \eta L)}{12LD_1}[-2h_0(h_0 + \eta L)(\log(h_0 + \eta L)$$
$$- \log(h_0)) + 2\eta h_0 L + \eta^2 L^2]$$

$$k_{44} = \frac{Eb(h_0 + \eta L)^2}{12LD_1}[2h_0^2(\log(h_0 + \eta L) - \log(h_0))$$
$$- 2\eta h_0 L + \eta^2 L^2] \quad (22)$$

*Case C*

$$k_{11} = \frac{4Ebs^3}{D_2 L^3}(s + \eta L)^3(5s^4 + 10\eta s^3 L + 10\eta^2 S^2 L^2$$
$$+ 5\eta^3 s L^3 + \eta^4 L^4)$$

$$k_{12} = -\frac{Ebs^4}{D_2 L^2}(s + \eta L)^3(10s^3 + 10\eta s^2 L + 5\eta^2 s L^2 + \eta^3 L^3)$$

$$k_{14} = -\frac{Ebs^3}{D_2 L^2}(s + \eta L)^4(10s^3 + 20\eta s^2 L$$
$$+ 15\eta^2 s L^2 + 4\eta^3 L^3)$$

$$k_{13} = -k_{11}; \quad k_{22} = \frac{2Ebs^5}{3D_2 L}(s + \eta L)^3(10s^2$$
$$+ 5\eta s L + \eta^2 L^2)$$

$$k_{23} = \frac{Ebs^4}{D_2 L^2}(s + \eta L)^3(10s^3 + 10\eta s^2 L + 5\eta^2 s L^2 + \eta^3 L^3)$$

$$k_{24} = \frac{Ebs^4}{D_2 L}(s + \eta L)^4(10s^2 + 10\eta s L + 3\eta^2 L^2)$$

$$k_{33} = k_{11} \; k_{34} = -k_{14}$$

$$k_{44} = \frac{2Ebs^3}{3D_2 L}(s + \eta L)^5(10s^2 + 15\eta s L + 6\eta^2 L^2) \quad (23)$$

## 5. Consistent Load Matrix

It is known that every element loading must be reduced to a set of nodal forces, and moreover, it is always possible to perform such a reduction on the basis of approximate physical intuition. For example, if the cross section variation is not too pronounced, it would be reasonable to use the nodal forces corresponding to the classical cubic element.

On the other hand, a set of *consistent* nodal forces can be easily obtained for every kind of element loading, by considering potential energy.

In this way, the nodal force vector **f** consistent with a distributed load $q$ acting between the arbitrary abscissae $a$ and $b$ can be written as:

$$\mathbf{f} = \int_a^b q(z)\mathbf{N}^{\mathrm{T}} \mathrm{d}z \quad (24)$$

For the sake of completeness, the nodal forces consistent with a uniformly distributed load $q$ along the entire element are:

*Case A*

$$f_1 = \frac{q}{6\beta D}[3\beta L(2b_0 - \beta L) - 2(3b_0^2 - \beta^2 L^2)[\log(b_0$$
$$+ \beta L) - \log(b_0)]]$$

$$f_2 = \frac{qL}{12D\beta}[2b_0(3b_0 + 2\beta L)[\log(b_0 + \beta L) - \log(b_0)]$$
$$- \beta L(6b_0 + \beta L)]$$

$$f_3 = \frac{q}{6D\beta}[2(3b_0^2 + 6\beta b_0 L + 2\beta^2 L^2)[\log(b_0 + \beta L)$$
$$- \log(b_0)] - 3\beta L(2b_0 + 3\beta L)]$$

$$f_4 = \frac{qL}{12D\beta}[2(3b_0^2 + 4\beta b_0 L + \beta^2 L^2)[\log(b_0 + \beta L)$$
$$- \log(b_0)] - \beta L(6b_0 + 5\beta L)]. \quad (25)$$

*Case B*

$$f_1 = \frac{q}{2\eta D_1}[\eta L(6h_0 + \eta L) - 2h_0(3h_0 + 2L\eta)[\log(h_0 + \eta L)$$
$$- \log(h_0)]]$$

$$f_2 = \frac{qh_0}{2\eta^3 LD_1}[\eta^2 L^2(2h_0 - \eta L) - \eta h_0 L(4h_0 + 3\eta L)$$
$$\times [\log(h_0 + \eta L) - \log(h_0)]$$
$$+ 2h_0(h_0 + \eta L)^2[\log(h_0 + \eta L)^2 + \log(h_0)^2]$$
$$- 4h_0(h_0 + \eta L)^2 \log(h_0 + \eta L)\log(h_0)]$$

$$f_3 = \frac{q}{2\eta D_1}[-\eta L(6h_0 + 5\eta L) + 2(3h_0^2 + 4\eta h_0 L + \eta^2 L^2)$$
$$\times [\log(h_0 + \eta L) - \log(h_0)]]$$

$$f_4 = \frac{q(h_0 + \eta L)}{2\eta^3 LD_1}[-\eta^2 L^2(2h_0 + 3\eta L)$$
$$+ \eta(4h_0^2 L + 5\eta h_0 L^2 + \eta^2 L^3)[\log(h_0 + \eta L) - \log(h_0)]$$
$$- 2h_0^2(h_0 + \eta L)[\log(h_0 + \eta L)^2 + \log(h_0)^2]$$
$$+ 4h_0^2(h_0 + \eta L)\log(h_0 + \eta L)\log(h_0)]. \quad (26)$$

Table 2
Vertical deflection at the midspan of a simply supported beam. First column: cases of cross section variation laws; second column: proposed method with two elements; third column: proposed method with four elements; fourth column: classical approach with four elements; fifth column: classical approach with 10 cubic hermitian elements; sixth column: classical approach with 100 cubic hermitian elements; seventh column; classical approach with 200 cubic hermitian elements; eighth column: classical approach with 400 cubic hermitian elements

| Case | Nonclassical | | Classical | | | | |
|---|---|---|---|---|---|---|---|
| | $n = 2$ | $n = 4$ | $n = 4$ | $n = 10$ | $n = 100$ | $n = 200$ | $n = 400$ |
| A | 3.73507 | 3.73507 | 4.02273 | 3.78396 | 3.73555 | 3.73519 | 3.73510 |
| B | 2.86824 | 2.86824 | 3.74803 | 3.16759 | 2.87143 | 2.86904 | 2.86844 |
| C | 5.11590 | 5.11590 | 7.22965 | 5.62399 | 5.12066 | 5.11709 | 5.11620 |

*Case C*

$$f_1 = \frac{qsL}{D_2}(10s^3 + 16\eta s^2 L + 8\eta^2 L^2 s + \eta^3 L^3)$$

$$f_2 = -\frac{qs^2 L^2}{6D_2}(10s^2 + 8\eta sL + \eta^2 L^2)$$

$$f_3 = \frac{qL}{D_2}(10s^4 + 24\eta s^3 L + 20\eta^2 L^2 s^2 + 7\eta^3 L^3 s + \eta^4 L^4)$$

$$f_4 = \frac{qL^2}{6D_2}(s + \eta L)^2(10s^2 + 12\eta sL + 3\eta^2 L^2).$$

## 6. Numerical Results

As a first example, let us consider a cantilever beam with span $L = 10$ m, subjected to an uniformly distributed load $q = 1$ $t\mathrm{m}^{-1}$. In Table 1 the vertical displacement at the free end is listed, for various cross section variation laws, and for various finite elements discretizations. The first row refers to a beam with unit depth and linearly varying width between 2 m at the clamped end and 0.25 m at the free end (Case A).

The second and third rows refer to a beam with unit width and varying depth between 2 m at the clamped end and 0.25 m at the free end. In the second row the depth is assumed to vary according to linear law (Case B), whereas in the third row the variation law is supposed to be quadratic (Case C). In all cases the Young's modulus is equal to $E = 300{,}000$ $t\mathrm{m}^{-2}$.

In the second column the vertical displacement is reported, obtained using a single finite element with varying cross section, the third and fourth columns contain the same displacement obtained using two and five finite elements of the same kind.

The fifth and sixth column give the vertical displacement obtained using three and 10 classical hermitian finite elements, respectively. The depth and the width of each cross sections were chosen to be equal to the middle values of the corresponding nonuniform element. Finally, the last two columns give the vertical displacement for a mesh with 100 and 200 constant finite elements, respectively.

The increase of the accuracy and the reduction of the computational effort is noticeable. As can be seen, a single nonclassical finite element turns out to be more precise than 200 classical cubic elements, and the predicted vertical displacement hardly changes if two or five nonclassical finite elements are employed.

In Table 2 the midspan vertical displacement is reported, for a simply supported beam with span $L = 10\,m$. The first row refers to a beam with unit depth and linearly varying width between 0.5 m at the supported ends and 2 m at the midspan (Case A).

The second and third rows refer to a beam with unit width and varying depth between 0.5 m at the supported ends and 2 m at the midspan. In the second row the depth is assumed to vary according to linear law (Case B), whereas in the third row the variation law is supposed to be quadratic (Case C). In all cases the Young's modulus is equal to $E = 300{,}000$ $t$ $\mathrm{m}^{-2}$. The columns description is the same as in the first example.

In this case the precision and the computational advantages of the proposed elements are even more evident. In fact, there is no difference between the vertical displacement predicted by a mesh with two nonclassical finite elements and the same displacement calculated employing four nonclassical elements. On the other hand, 400 classical cubic finite elements were necessary, in order to obtain an accurate solution.

## 7. Conclusion

Three nonclassical finite elements have been proposed, for the static analysis of Euler–Bernoulli beams with varying rectangular cross sections.

Nonpolynomial shape functions have been employed, using an exact solution given in Ref. [1]; the stiffness matrices have been calculated with the aid of a symbolic package. The equivalent nodal forces, consistent with an uniformly distributed load, have also been given. Numerical examples show that the use of these proposed finite elements can be competitive with the more usual cubic elements.

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