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# Refined semianalytical solutions in the approximations of higher eigenvalues

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**Abstract**

In this paper some vibration problems are solved by means of the optimized Rayleigh and Timoshenko quotients, and the first few eigenvalues are approximated. The iterative method of Ku is also used in order to deduce close lower–upper bounds, according to the Ly formula. Extensive use of symbolic language allows us to obtain closer approximations than the usual ones, because multiple–parameter quotients can be employed, and more than one iteration can be performed.

Two archetypical examples will be used throughout the paper, i.e. the truncated and complete wedge beams, and constant reference will be made to available results from the literature. 

*Keywords:* Dynamics; Stability; Rayleigh quotient; Eigenvalues optimization

## 1. Introduction

Critical loads and free vibration frequencies of beams subjected to axial loads must be often calculated, at least as a preliminary step toward a bifurcation analysis. The problem can be generally reduced to a linear eigenvalue problem, and it is necessary to enlighten the influence of various structural control parameters on the eigensolutions, so that coincident eigenvalues or coalescing frequencies can be detected.

All the analytical solution methods are ideally suited to this end, but they are bounded by the well-known unavailability of general methods for solving linear equations with nonconstant coefficients. Actually, the range of analytical solutions is limited to few variation laws of the cross-section and of axial load distributions, so that these solutions can be only used as benchmark for more general approaches.

Numerical methods, on the contrary, have a general range of applicability, and can virtually solve every stability or dynamic problem. Nevertheless, if it is necessary to perform a parametric study, then it is necessary to perform a large sample of numerical simulations, which in turn can only give an approximate answer to the problem at hand.

The semianalytical method (henceforth SAN method) seems to share the advantage of both the analytical and numerical methods. In fact, their range of applicability is the same as for the numerical methods, yet a parametric study is as simple as in the analytical approaches. Moreover, the potentialities of the SAN methods have been tremendously enlarged by the widespread availability of symbolic languages on personal computers, so that it is not difficult to predict a renewed interest in the SAN methods.

## 2. Refined SAN Methods

The first approximate formula for the free vibration frequencies goes back to Rayleigh [1], and his approach is still adopted in numerous engineering fields, as structural engineering, mechanical engineering, seismic analysis. According to the Rayleigh quotient, an upper bound to the first frequency is obtained,

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and its closeness to the true result is strongly influenced by the chosen trial function.

An improved version of the Rayleigh quotient was given by the same author in 1894, [2], who used a trial function defined by a polynomial with the noninteger undetermined power (the so-called *non-integer power Rayleigh method*). In this way a nonlinear optimization problem is generated, whose solution can be usually reached only with numerical methods.

Although the optimized Rayleigh quotient was successfully used by Stodola in 1927 [3], the intrinsic mathematical difficulties led to neglect this powerful approach, which was recently re-discovered by Schmidt and Bert [4], [5], and consequently the non-integer power Rayleigh method is now known as the *Rayleigh–Schmidt* method.

Later on, a different implementation of the same quotient was proposed by Elishakoff [6], in which an undetermined multiplier rather than un undetermined power is used (*non-integer multiplier Rayleigh method*). It seems that this choice lead to simpler formulae, and even to more accurate results [7], [8].

A dual approach to the Rayleigh quotient was proposed by Timoshenko [9], who used the total complementary energy instead of the total potential energy, in order to obtain an approximation to the critical loads. In this way another quotient can be defined, which is always an upper bound to the true result, and it turns out to be always not greater than the corresponding Rayleigh quotient [10].

However, the original Timoshenko quotient was readily usable only for statically determinate structures, where the bending moment can be immediately calculated. More recently, the method was extended to redundant structures, to general three-dimensional continuum problems, and to dynamic analysis [11], [12].

Finally, it should be noted that the Timoshenko quotient can be optimized exactly in the same way as the Rayleigh quotient, giving rise to the Timoshenko–Schmidt quotient.

A totally different improved version of the Rayleigh and Timoshenko quotient was proposed by Ku [10]. According to this method, a (usually simple) trial function is used as starting point for an iterative procedure which generate a sequence of decreasing upper bounds.

This approach has been originally proposed in the stability analysis of uniform or tapered beams, and has been subsequently applied to frequency analysis and stepped beams. Its convergence rate is usually quite satisfactory, and from an engineering point of view, the first few iterations lead to acceptable results.

Both the above-mentioned refined methods furnish upper bounds, while it is often necessary to obtain even a close lower bound, at least in the stability analysis. Basically, all the lower bound formulae can be divided into two groups; the first one goes back to Temple [13], and requires the exact knowledge of the second eigenvalue, the other one is based on the values of the Timoshenko and Rayleigh quotients and is extremely more manageable, both from a numerical and an analytical point of view.

The first lower bound formula of the second group seems to be the Shreyer–Shih formula [12], which has been subsequently used by Popelar and improved by Ku and Hanna and Michalopoulos [14]. More recently, a substantially new approach was used by Schmidt [15], [16] and Ly [17], who proposed very satisfactory lower bound formulae.

Finally, it is important to stress that all the formulae of the second group lead to close lower bounds, if they are used hand by hand with a refined SAN method.

In this paper a powerful symbolic language [18] is used to obtain very accurate approximations to some vibrations problems. The use of symbolic software allowed us to apply the non-integer multiplier Rayleigh method with *two* or more undetermined multipliers, so that a convergence curve can be sketched.

## 3. A Convergence Curve

As a first example, let us examine the dynamic behaviour of a propped cantilever with rectangular cross-section and linearly varying height, (Fig. 1), so that cross-sectional area and second moment of area are given by:

$$A(z) = A_0\left(1 + c\frac{z}{l}\right) \quad I(z) = I_0\left(1 + c\frac{z}{l}\right)^3, \tag{1}$$

respectively, where $l$ is the span of the beam, and $c$ is the taper ratio.

The differential equation of motion of this beam can be solved in terms of Bessel functions [19], but the resulting eigenvalue problem is quite tedious to solve numerically.

On the contrary, the optimized SAN methods quickly give a close upper bound to the true frequency. The Rayleigh quotient and Timoshenko quotient can be written as:

$$\omega_R^2 = \int_0^l EIv''^2(z)\,dz \Big/ \int_0^l \rho A v^2(z)\,dz, \tag{2}$$

and

$$\omega_T^2 = \int_0^l \rho A v^2(z)\,dz \Big/ \int_0^l \frac{m^2(z)}{EI}\,dz, \tag{3}$$

respectively, where $E$ is the Young modulus, $\rho$ is the mass density and $m(z)$ is an admissible *moment* function [10].

The crucial step is the choice of the trial function $v(z)$, which must at least satisfy the geometrical bound-

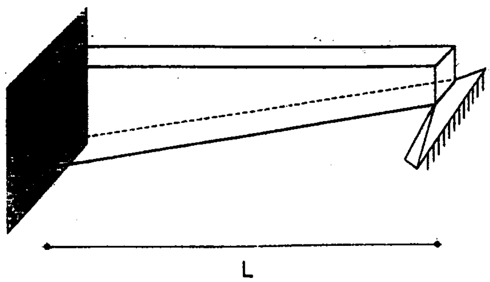


Fig. 1. Propped cantilever beam with linearly varying height.

ary conditions. However, the underlying philosophy of the refined SAN method allows the choice of the simplest trial function, which is subsequently refined as many times as necessary. An iterative procedure which allows this kind of refinement has been proposed in [20].

In this case, we start from the function:

$$v(z) = z^2(l - z), \tag{4}$$

which obviously satisfies the conditions:

$$v(0) = v'(0) = v(l) = 0. \tag{5}$$

If $c = -9/10$, a classical approach gives an approximate non-dimensional frequency equal to:

$$R = \frac{\rho A}{EI}\omega_R^2 l^4 = 108.528, \tag{6}$$

if the Rayleigh quotient is used, whereas the corresponding Timoshenko approach behaves much better, and results in a closer upper bound to the true frequency:

$$T = \frac{\rho A}{EI}\omega_T^2 l^4 = 78.3470. \tag{7}$$

A lower bound can be immediately deduced, by using for example the recently proposed Ly formula [17]:

$$L = R\left(\frac{T}{R} - 2\frac{T^2}{R^2} + 2\frac{T^3}{R^3}\right)^{1/2}, \tag{8}$$

and turns out to be equal to:

$$L = 71.33596. \tag{9}$$

The exact value is equal to 74.4272.

A systematic optimization procedure is then based on the trial functions:

$$v_n(z) = z^2(l - z)\left(1 + \sum_{i=1}^{n} t_i z^i\right). \tag{10}$$

For each $n$, the following equations:

$$\frac{\partial R}{\partial t_i} = 0 \quad i = 1, \ldots n, \tag{11}$$

must be solved, in order to find the unknown multipliers $t_i$, and then the optimized Rayleigh quotient can be immediately found. A similar procedure can be used for the Timoshenko quotient, and the convergence curve as in Fig. 2 can be sketched.

In the same figure the convergence curve of the lower bound is also given, as obtained by using the Ly formula. The good performance of the Timoshenko quotient can be easily observed, whereas the Rayleigh

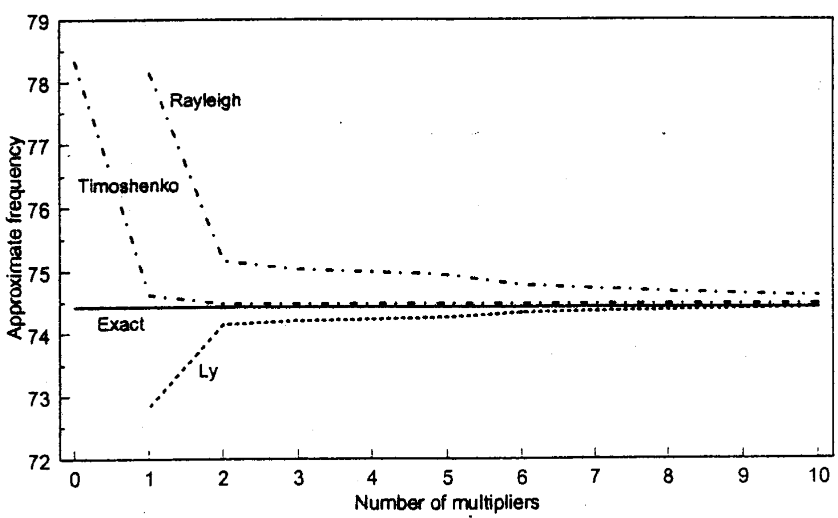


Fig. 2. Convergence curve for a tapered propped cantilever beam: approximate.

quotient shows a sluggish convergence to the true result.

## 4. The Truncated Wedge Beam

The optimized quotients can also be used to approximate higher eigenvalues, as indicated in [21] using a Galerkin approach. In the following, a similar procedure will be adopted in order to obtain upper bounds for the first four frequencies of a cantilever beam with linearly varying breadth and constant height (Fig. 3).

More precisely, if $c$ is the ratio between the breadth at the right end and the breadth at the left end, then area and second moment of area will vary according to the laws:

$$A(z) = A_0\left(1 + \frac{1-c}{c}\frac{z}{l}\right) \quad I(z) = I_0\left(1 + \frac{1-c}{c}\frac{z}{l}\right), \tag{12}$$

and $A_0$, $I_0$ are the cross sectional area and inertia at the left end, respectively.

This case was treated in detail by Naguleswaran [21], and its results can be considered exact, and therefore used as benchmarks.

On the other hand, we shall employ optimized versions of Rayleigh–Ritz and Timoshenko–Ritz methods, with one unknown multiplier, so that upper bounds to the true results will be obtained.

The following trial functions will be used:

$$v(z) = (a_1 z^2 + a_2 z^3 + a_3 z^4 + a_4 z^5)(1 + t_1 z), \tag{13}$$

where $a_i$ are the Ritz coefficients, and $t$ is the optimization multiplier.

If the Rayleigh quotient (2) is employed, then the following eigenvalue problem is obtained:

$$|k - \omega^2 m| = 0, \tag{14}$$

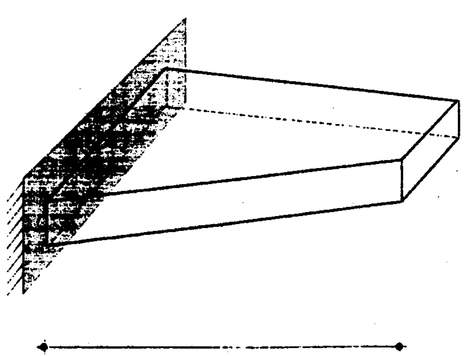


Fig. 3. Cantilever beam with linearly varying breadth.

where the entries of the (symmetric) stiffness matrix **k** and mass matrix **m** can be obtained by means of a straightforward application of the Castigliano theorem:

$$k_{11} = \frac{2(2 + 2c + 8t_1 + 4ct_1 + 9t_1^2 + 3ct_1^2)}{c}, \tag{15}$$

$$k_{12} = \frac{2(20 + 10c + 75t_1 + 25ct_1 + 72t_1^2 + 18ct_1^2)}{5c}, \tag{16}$$

$$k_{13} = \frac{4(15 + 5c + 56t_1 + 14ct_1 + 50t_1^2 + 10ct_1^2)}{5c}, \tag{17}$$

$$k_{14} = \frac{4(28 + 7c + 105t_1 + 21ct_1 + 90t_1^2 + 15ct_1^2)}{7c}, \tag{18}$$

$$k_{22} = \frac{6(15 + 5c + 48t_1 + 12ct_1 + 40t_1^2 + 8ct_1^2)}{5c}, \tag{19}$$

$$k_{23} = \frac{4(252 + 63c + 770t_1 + 154ct_1 + 600t_1^2 + 100ct_1^2)}{35c}. \tag{20}$$

$$k_{24} = \frac{2(140 + 28c + 20t_1 + 70ct_1 + 315t_1^2 + 45ct_1^2),}{7c} \tag{21}$$

$$k_{33} = \frac{4(420 + 84c + 1200t_1 + 200ct_1 + 875t_1^2 + 125ct_1^2)}{35c}, \tag{22}$$

$$k_{34} = \frac{10(144 + 24c + 399t_1 + 57ct_1 + 280t_1^2 + 35ct_1^2)}{21c} \tag{23}$$

$$k_{44} = \frac{20(105 + 15c + 280t_1 + 35ct_1 + 189t_1^2 + 21ct_1^2)}{21c}, \tag{24}$$

and:

$$m_{11} = \frac{1}{15} + \frac{1}{3c} + \frac{2t_1}{21} + \frac{4t_1}{7c} + \frac{t_1^2}{28} + \frac{t_1^2}{4c}, \tag{25}$$

$$m_{12} \frac{72 + 12c + 126t_1 + 18ct_1 + 56t_1^2 + 7ct_1^2}{252c}, \tag{26}$$

$$m_{13} = \frac{1}{28} + \frac{1}{4c} + \frac{t_1}{18} + \frac{4t_1}{9c} + \frac{t_1^2}{45} + \frac{t_1^2}{5c}, \tag{27}$$

$$m_{14} = \frac{440 + 55c + 792t_1 + 88ct_1 + 360t_1^2 + 36ct_1^2}{1980c}, \tag{28}$$

$$m_{22} = m_{13} \quad m_{23} = m_{14}, \tag{29}$$

$$m_{24} = \frac{198 + 22c + 360t_1 + 36ct_1 + 165t_1^2 + 15ct_1^2}{990c}, \quad (30)$$

$$m_{33} = m_{24}, \quad (31)$$

$$m_{34} = \frac{1}{55} + \frac{2}{11c} + \frac{t_1}{33} + \frac{t_1}{3c} + \frac{t_1^2}{78} + \frac{2t_1^2}{13c}, \quad (32)$$

$$m_{44} = \frac{1}{66} + \frac{1}{6c} + \frac{t_1}{39} + \frac{4t_1}{13c} + \frac{t_1^2}{91} + \frac{t_1^2}{7c}. \quad (33)$$

If the Timoshenko approach is preferred, then the application of Eq. (3) leads to the same eigenvalue problem, but now the entries of the stiffness matrix look more complicated, and are not given here.

However, both above-mentioned procedures can be easily translated into a symbolic program. For example, in the Appendix 2 simple *Mathematica* notebooks are reported, which can generate the matrices, calculate the eigenvalues and solve the optimization problem.

In both the cases, the four (real) eigenvalues $\omega_i^2$ of the symmetrizable problem (14) are functions of the unknown multiplier $t_1$ and can be optimized with respect to it, by imposing:

$$\frac{\partial \omega_1^2}{\partial t_1} = 0. \quad (34)$$

It is perhaps worth noting that other boundary conditions can be easily treated, modifying the second row of the notebooks, whereas other variation laws of the cross section can be dealt with by modifying the third row, where area and inertia are defined.

In Table 1 the first four non-dimensional frequencies:

$$\Omega_i = \sqrt{\frac{\rho A(l)\omega_i^2 l^4}{EI(l)}}. \quad (35)$$

have been reported, as obtained with the Rayleigh approach and with the Timoshenko approach. In both cases, the values of the unknown multiplier has been given in brackets.

For the sake of comparison, the first three frequencies are also given, as deduced from Naguleswaran [22]. It is immediately seen the excellent performances of the Timoshenko method, whereas the Rayleigh quotient is less satisfactory, at least for the higher eigenvalues.

Another approach which allows a substantial refinement of the eigenvalues goes back to Ku [10], who employed an iterative method to approximate the first critical load of beams. The same approach was also used to obtain the first frequency [12], and it will be here generalized to give refined upper bounds even for the higher eigenvalues.

Let us consider a proposed cantilever beam with varying area and inertia according to Eq. (12), and let us use the following trial functions:

$$v_1(z) = \frac{z^2(3l^2 - 5lz + 2z^2)}{2l^4}, \quad (36)$$

$$v_2(z) = \frac{z^2(-5l^3 + 19l^2 z - 22lz^2 + 8z^3)}{8l^5}, \quad (37)$$

which both satisfy all the boundary conditions. From the assumed shape function $v(z) = a_1 v_1(z) + a_2 v_2(z)$ it is necessary to deduce an admissible bending moment, by imposing an equilibrium condition and a compatibility condition, as indicated by Ku.

After some algebra, the resulting moment is given by:

$$m(z) = -\frac{A(l-z)^3}{3360cl^6} + (lz - l^2)\frac{B}{C}. \quad (38)$$

where:

$$\begin{aligned} A = {} & l^5(88a_1 + 52ca_1 + 10a_2 + ca_2) \\ & + l^4 z(124a_1 + 44ca_1 + 19a_2 + 7ca_2) \\ & + l^3 z^2(108a_1 - 24ca_1 + 27a_2 + 18ca_2) \\ & + l^2 z^3(40a_1 - 152ca_1 + 34a_2 + 34ca_2) \\ & + lz^4(-80a_1 + 80ca_1 + 40a_2 - 120ca_2) \\ & + 60a_2 z^5(c - 1), \end{aligned}$$

$$\begin{aligned} B = {} & -65032a_1 + 416656ca_1 - 1101324c^2 a_1 \\ & + 1513120c^3 a_1 - 1116220c^4 a_1 \\ & + 361032c^5 a_1 + 66444c^6 a_1 - 110976c^7 a_1 \\ & + 42180c^8 a_1 - 5880c^9 a_1 - 6620a_2 \\ & + 48532ca_2 - 152187c^2 + 260680c^3 a_2 \\ & - 243775c^4 a_2 + 123970c^5 a_2 - 33957c^6 a_2 \\ & + 2112c^7 a_2 + 1595c^8 a_2 - 350c^9 a_2 + 36960a_2 \log(l) \\ & - 178080ca_1 \log(l) + 317520c^2 a_1 \log(l) \\ & - 235200c^3 a_1 \log(l) + 58800c^4 a_1 \log(l) \\ & + 4200a_2 \log(l) - 24360ca_2 \log(l) \\ & + 55860c^2 a_2 \log(l) - 58800c^3 a_2 \log(l) \\ & + 14700c^4 a_2 \log(l) - 36960a_1 \log(cl) \\ & + 178080ca_1 \log(cl) - 317520c^2 a_1 \log(cl) \\ & + 235200c^3 a_1 \log(cl) - 58800c^4 a_1 \log(cl) \\ & - 4200a_2 \log(cl) + 24360ca_2 \log(cl) \\ & - 55860c^2 a_2 \log(cl) + 58800c^3 a_2 \log(cl) \\ & - 14700c^4 a_2 \log(cl), \end{aligned}$$

$$C = 705600(c - 1)^6 c(3 - 4c + c^2 - 2\log(l) + 2\log(cl)). \quad (41)$$

Table 1
First four non-dimensional frequencies for a cantilever truncated wedge beam

| $c$ | $\Omega_i$ | Rayleigh–Ritz | Timoshenko–Ritz | Naguleswaran |
|---|---|---|---|---|
| 0.05 | $\Omega_1$ | 1.58741 (−0.407194) | 1.54561 (−0.629215) | 1.5456 |
| | $\Omega_2$ | 17.2118 (−0.669764) | 16.9963 (−0.771774) | 16.9955 |
| | $\Omega_3$ | 57.0052 (−0.687538) | 55.7997 (−0.734306) | 55.7660 |
| | $\Omega_4$ | 127.056 (−0.880641) | 114.778 (−0.957276) | |
| 0.1 | $\Omega_1$ | 1.82404 (−0.381452) | 1.81132 (−0.437251) | 1.8113 |
| | $\Omega_2$ | 17.9453 (−0.618561) | 17.8806 (−0.720183) | 17.8803 |
| | $\Omega_3$ | 57.7832 (−0.683023) | 57.1402 (−0.715843) | 57.1359 |
| | $\Omega_4$ | 127.042 (−0.879964) | 116.350 (−0,952997) | |
| 0.15 | $\Omega_1$ | 2.01316 (−0.357308) | 2.008364 (−0.437251) | 2.0084 |
| | $\Omega_2$ | 18.5107 (−0.523559) | 18.4885 (−0.659681) | 18.4884 |
| | $\Omega_3$ | 58.4557 (−0.678802) | 57.9740 (−0.772651) | 57.9685 |
| | $\Omega_4$ | 127.0603 (−0.87933) | 117.2840 (−0.949398) | |
| 0.20 | $\Omega_1$ | 2.17288 (−0.333606) | 2.17086 (−0.304404) | 2.1709 |
| | $\Omega_2$ | 18.9668 (−0.235934) | 18.9589 (−0.570098) | 18.9589 |
| | $\Omega_3$ | 59.0452 (−0.674876) | 58.5687 (−0.819771) | 58.5574 |
| | $\Omega_4$ | 127.101 (−0.878731) | 117.940 (−0.946183) | |
| 0.25 | $\Omega_1$ | 2.31237 (−0.309274) | 2.31146 (−0.304404) | 2.3115 |
| | $\Omega_2$ | 19.3473 (0.0022311) | 19.3445 (−0.140196) | 19.3445 |
| | $\Omega_3$ | 59.5679 (−0.671216) | 59.0249 (−0.837547) | 59.0079 |
| | $\Omega_4$ | 127.158 (−0.878162) | 118.444 (−0.943235) | |
| 0.30 | $\Omega_1$ | 2.43696 (−0.283772) | 2.43653 (−0.329689) | 2.4365 |
| | $\Omega_2$ | 19.6727 (0.177421) | 19.6718 (−0.433603) | 19.6717 |
| | $\Omega_3$ | 60.0361 (−0.667814) | 59.3924 (−0.848707) | 59.3704 |
| | $\Omega_4$ | 127.227 −0.877619) | 118.851 (−0.94045) | |
| 0.35 | $\Omega_1$ | 2.55002 (−0.25204) | 2.54981 (−0.262513) | 2.5498 |
| | $\Omega_2$ | 19.9565 (0.321097) | 19.9562 (−0.479058) | 19.9561 |
| | $\Omega_3$ | 60.4590 (−0.664572) | 59.6988 (−0.856175) | 59.6727 |
| | $\Omega_4$ | 60.4590 (−0.664572) | 119.194 (−0.937813) | |
| 0.40 | $\Omega_1$ | 2.65383 (−0.21454) | 2.65372 (−0.262267) | 2.6537 |
| | $\Omega_2$ | 20.2079 (0.444322) | 20.2078 (−0.508877) | 20.2077 |
| | $\Omega_3$ | 60.8399 (−0.58824) | 59.9612 (−0.761392) | 59.9317 |
| | $\Omega_4$ | 127.389 (−0.8766) | 119.491 (−0.935339) | |
| 0.45 | $\Omega_1$ | 2.74500 (−0.156375) | 2.74994 (−0.196222) | 2.7499 |
| | $\Omega_2$ | 20.43349 (0.552142) | 20.4334 (−0.526794) | 20.4332 |
| | $\Omega_3$ | 61.1733 (−0.532459) | 60.1905 (−0.864426) | 60.1582 |
| | $\Omega_4$ | 127.477 (−0.876116) | 119.752 (−0.932823) | |
| 0.50 | $\Omega_1$ | 2.83972 (−0.0918137) | 2.83969 (−0.0894527) | 2.8397 |
| | $\Omega_2$ | 20.7020 (−0.466242) | 20.6379 (−0.53725) | 20.6376 |
| | $\Omega_3$ | 61.4671 (−0.48793) | 60.3941 (−0.869043) | 60.3597 |
| | $\Omega_4$ | 127.568 (−0.875651) | 119.981 (−0.931467) | |
| 0.55 | $\Omega_1$ | 2.92390 (−0.0349454) | 2.92389 (−0.0894527) | 2.9239 |
| | $\Omega_2$ | 20.8957 (−0.480278) | 20.8248 (−0.545327) | 20.8245 |
| | $\Omega_3$ | 61.7283 (−0.448311) | 60.5748 (−0.87497) | 60.5414 |
| | $\Omega_4$ | 127.662 (−0.875201) | 120.168 (−0.930265) | |
| 0.60 | $\Omega_1$ | 3.00327 (0.00775039) | 3.00326 (−0.0894527) | 3.0033 |
| | $\Omega_2$ | 21.0732 (−0.490341) | 20.9971 (−0.537744) | 20.9967 |
| | $\Omega_3$ | 61.9623 (−0.412059) | 60.7244 (−0.907011) | 60.7072 |
| | $\Omega_4$ | 127.757 (−0.874765) | 119.825 (−0.931096) | |

From this, it is now possible to calculate the strain energy:

$$L = \frac{1}{2}\int_0^l \frac{m^2(z)}{EI}\,dz, \tag{42}$$

and the kinetic energy:

$$T = \frac{1}{2}\int_0^l \rho A v^2\,dz. \tag{43}$$

The stiffness matrix and the mass matrix can be immediately deduced, and the resulting eigenvalue problem can be solved to give the first two approximate frequencies. If a refinement is needed, the moment can be integrated twice, and a new trial function is obtained, which in turn generate a new admissible moment, a refined strain energy and kinetic energy, and a second approximation of the first two frequencies.

In principle, the procedure can be repeated as needed, but usually two iterations can lead to very complicated formulae, which become unmanageable.

For the sake of comparison, in Table 2 we have reported the exact results, as given by Naguleswaran, the first two iterations of the Ku approach, and the approximate values as given by a Rayleigh–Ritz optimized method with two undetermined multipliers.

In this latter case, the following trial function has been used:

$$v(z) = (a_1 v_1 + a_2 v_2)(1 + t_1 z + t_2 z^2). \tag{44}$$

It is worth observing that the Ku method is extremely prone to numerical errors, so that exact calculations must be carried out, or a large number of significant figures must be used (up to 32 bit precision). On the other hand, it is faster than the two parameter Rayleigh optimization approach.

## 5. The Complete Beam

A particularly intriguing case is given by the so-called 'complete beam', in which the cross section vanishes at one end. In this case the sharp end must be free, because it cannot sustain any bending moment or shear force, and the only significant structural system is the cantilever beam.

Moreover, some numerical and analytical approaches tend to give wrong results, because of numerical instabilities associated to the integrand behaviour of the energies near the sharp end. It is therefore interesting to compare some approximate results with the benchmark reported in [22].

In Table 3 a large set of SAN results have been given and compared with the exact results. More precisely, in the first row the first two eigenvalues are written down, as obtained by means of a one-parameter Rayleigh optimization approach and trial function:

$$v(z) = (a_1 z^2 + a_2 z^3)(1 + t_1 z). \tag{45}$$

The following three rows indicate the improvements which can be obtained by using an increasing number of unknown multipliers. The trial functions are equal to:

$$v(z) = (a_1 z^2 + a_2 z^3)(1 + t_1 z + t_2 z^2), \tag{46}$$

Table 2
First two non-dimensional frequencies for a propped cantilever truncated wedge beam

| $c$ | $\Omega_i$ | Rayleigh | Ku 1st iter. | Ku 2nd iter. | Naguleswaran |
|---|---|---|---|---|---|
| 0.05 | $\Omega_1$ | 12.2166 | 12.0749 | 12.0712 | 12.0698 |
| | $\Omega_2$ | 45.6785 | 45.9425 | 45.4684 | 45.3869 |
| 0.1 | $\Omega_1$ | 12.7579 | 12.7163 | 12.7155 | 12.7143 |
| | $\Omega_2$ | 46.6172 | 46.9248 | 46.6618 | 46.5654 |
| 0.15 | $\Omega_1$ | 13.1665 | 13.1521 | 13.1521 | 13.1512 |
| | $\Omega_2$ | 47.3245 | 37.5247 | 47.3649 | 47.2743 |
| 0.2 | $\Omega_1$ | 13.4903 | 13.4848 | 13.4852 | 13.4845 |
| | $\Omega_2$ | 47.8718 | 47.9484 | 47.8467 | 47.7672 |
| 0.25 | $\Omega_1$ | 13.7558 | 13.7536 | 13.7539 | 13.7535 |
| | $\Omega_2$ | 48.3046 | 48.2708 | 48.2038 | 48.1361 |
| 0.3 | $\Omega_1$ | 13.9790 | 13.9781 | 13.9784 | 13.9781 |
| | $\Omega_2$ | 48.6530 | 48.5279 | 48.4821 | 48.4255 |
| 0.35 | $\Omega_1$ | 14.1705 | 14.1699 | 14.1701 | 14.1699 |
| | $\Omega_2$ | 48.9379 | 48.7399 | 48.7069 | 48.6601 |
| 0.4 | $\Omega_1$ | 14.3373 | 14.3367 | 14.3367 | 14.3362 |
| | $\Omega_2$ | 49.1741 | 48.9192 | 48.8933 | 48.8550 |
| 0.45 | $\Omega_1$ | 14.4843 | 14.4834 | 14.4834 | 14.4832 |
| | $\Omega_2$ | 49.3721 | 49.0738 | 49.0513 | 49.0201 |
| 0.5 | $\Omega_1$ | 14.6149 | 14.6138 | 14.6137 | 14.6136 |
| | $\Omega_2$ | 49.5399 | 49.2092 | 49.1874 | 49.1622 |
| 0.55 | $\Omega_1$ | 14.7320 | 14.7308 | 14.7306 | 14.7301 |
| | $\Omega_2$ | 49.6834 | 49.3293 | 49.3064 | 49.2861 |
| 0.6 | $\Omega_1$ | 14.8375 | 14.8364 | 14.8361 | 14.8342 |
| | $\Omega_2$ | 49.9148 | 49.4370 | 49.4115 | 49.3955 |

$$v(z) = (a_1z^2 + a_2z^3)(1 + t_1z + t_2z^2 + t_3z^3); \qquad (47)$$

$$v(z) = (a_1z^2 + a_2z^3)(1 + t_1z + t_2z^2 + t_3z^3 + t_4z^4), \qquad (48)$$

respectively.

It is also possible to use the Chebyshev polynomial, instead of the power series, so obtaining the trial functions:

$$v(z) = (a_1z^2 + a_2z^3)\left(1 + \sum_{i=1}^{n} t_iT_i(z)\right), \qquad (49)$$

if the Chebyshev polynomials of the first kind are used, or:

$$v(z) = (a_1z^2 + a_2z^3)\left(1 + \sum_{i=1}^{n} t_iU_i(z)\right), \qquad (50)$$

if the Chebyshev polynomials of the second kind are employed.

In the next six rows the first two eigenvalues are reported, as obtained for $n = 1$, $n = 2$ and $n = 3$, and employing in turn the $T_i$ and the $U_i$ polynomials. It is interesting to note that, in this case, the Chebyshev polynomials of the first kind always behave better than the Chebyshev polynomials of the second kind.

Another improvement can be obtained by approximating more eigenvalues. For example, in the next row the first three eigenvalues have been approximated by using a Timoshenko–Ritz approach with a single unknown multiplier and the trial function:

$$v(z) = (a_1z^2 + a_2z^3 + a_3z^4)(1 + t_1z). \qquad (51)$$

The stiffness matrix will be given by:

$$k_{11} = \frac{64548 + 90628t_1 + 31897t_1^2}{49896000},$$
$$k_{12} = \frac{589082 + 850395t_1 + 307342t_1^2}{648648000}, \qquad (52)$$

$$k_{13} = \frac{1525069 + 2248512t_1 + 828545t_1^2}{2270268000}, \qquad (53)$$

$$k_{22} = \frac{20318389 + 30119516t_1 + 11179576t_1^2}{31783752000}, \qquad (54)$$

$$k_{23} = \frac{7529879 + 11389603t_1 + 4310780t_1^2}{15891876000}, \qquad (55)$$

$$k_{33} = \frac{89436608 + 137944960t_1 + 53242925t_1^2}{254270016000}, \qquad (56)$$

and the mass matrix can be written as:

$$m_{11} = \frac{1}{15} + \frac{2t_1}{21} + \frac{t_1^2}{28}, \quad m_{12} = \frac{1}{21} + \frac{t_1}{14} + \frac{t_1^2}{36},$$
$$m_{13} = \frac{1}{28} + \frac{t_1}{18} + \frac{t_1^2}{45}, \qquad (57)$$

Table 3
First four non-dimensional frequencies for a cantilever complete wedge beam, as obtained with a variety of approximate SAN methods

| Method | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ |
|---|---|---|---|---|
| Rayleigh 1 parameter | 7.15864 | 31.2576 | | |
| Rayleigh 2 parameters | 7.15791 | 31.1613 | | |
| Rayleigh 3 parameters | 7.15657 | 31.0587 | | |
| Rayleigh 4 parameters | 7.15647 | 31.0462 | | |
| Rayleigh 2 parameters $T_i$ polynomials | 7.15791 | 31.1613 | | |
| Rayleigh 3 parameters $T_i$ polynomials | 7.15653 | 31.0455 | | |
| Rayleigh 4 parameters $T_i$ polynomials | 7.15647 | 31.0419 | | |
| Rayleigh 2 parameters $U_i$ polynomials | 7.15791 | 31.1612 | | |
| Rayleigh 3 parameters $U_i$ polynomials | 7.15655 | 31.0597 | | |
| Rayleigh 4 parameters $U_i$ polynomials | 7.15647 | 31.0424 | | |
| Timoshenko 1 parameter | 7.15646 | 31.0427 | 75.7276 | |
| Timoshenko 1 parameter | 7.15646 | 31.0415 | 75.5264 | 141.1265 |
| Timoshenko 2 parameters | 7.15646 | 31.0413 | 75.5242 | 139.9510 |
| Naguleswaran | 7.15646 | 31.0413 | 75.4866 | 139.6100 |

$$m_{22} = \frac{1}{28} + \frac{t_1}{18} + \frac{t_1^2}{45}, \quad m_{23} = \frac{1}{36} + \frac{2t_1}{45} + \frac{t_1^2}{55},$$
$$m_{33} = \frac{1}{45} + \frac{2t_1}{55} + \frac{t_1^2}{66}. \tag{58}$$

The results show a good agreement, even for the third eigenvalue. A further improvement can be considered by approximating even the fourth eigenvalue, with the trial function:

$$v(z) = (a_1 z^2 + a_2 z^3 + a_3 z^4 + a_4 z^5)(1 + t_1 z). \tag{59}$$

The results are given in the next row, and obviously are quite satisfactory. Finally, a two-parameter version of the Timoshenko–Ritz method has been used, and the first four eigenvalues have been calculated, with the aid of the trial function:

$$v(z) = (a_1 z^2 + a_2 z^3 + a_3 z^4 + a_4 z^5)(1 + t_1 z + t_2 z^2). \tag{60}$$

The results, as given in the last row, should represent the best approximation to the true frequencies.

## 6. Conclusions

In this paper the optimized versions of the Rayleigh quotient and Timoshenko quotient have been used to approximate higher frequencies, and some procedures to systematically improve the approximations have been indicated. All the calculations have been performed with the aid of a powerful symbolic language, and two sample notebooks have been proposed in the Appendix. Constant references have been made to the wedge beam, which can be considered as a classical example.

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