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PLATE BENDING ANALYSIS BY THE CELL METHOD: NUMERICAL COMPARISONS WITH FINITE ELEMENT METHODS

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Abstract—A recently developed discretization method is applied to some classical problems in plate bending analysis, in order to check the accuracy of the solutions and the rate of convergence for various discretization levels. Numerical comparisons with available finite element solutions are reported, and it is shown that the proposed method can be considered competitive with the more recent finite element techniques.

1. INTRODUCTION

Consider the rectangular plate shown in Fig. 1, with arbitrary boundary conditions, and subjected to a general ensemble of transverse forces, F. The plate thickness, t, can vary according to a generic—even discontinuous—law, whereas the Young's modulus, E, and the Poisson ratio, ν, are assumed to be constant.

The structure can be discretized according to the finite element method, and countless elements have been used to this end. A useful review of these developments was completed by Hrabok and Hrudey [1]. More recently, nonconforming transition plate bending elements have been introduced [2], to refine the mesh of the plate locally, and a high-precision element having linearly varying thickness was used to study plates with varying flexural rigidity [3].

In this paper, the cell discretization method—which was recently extended to two-dimensional structures [4]—is used to examine the static behaviour of the plate in Fig. 1. A number of classical boundary conditions and load conditions are considered, which were already used to test the convergence properties of new finite elements.

2. METHOD OF ANALYSIS

The horizontal side length, L, is divided into t_1 small segments, while the vertical side length, H, is divided into t_2 small segments. Consequently, the plate is divided into $(t_1 t_2)$ rectangular panels. According to the cell discretization method, the panels are considered to be flexurally rigid and torsionally elastic, with the bending flexibilities being concentrated at the $(t_1 + 1)(t_2 + 1)$ vertices. It is possible to prove that the vertical displacements of these vertices can be assumed to be Lagrangian coordinates, so that the structure is reduced to an n-degree-of-freedom system, where $n = (t_1 + 1)(t_2 + 1)$. If the Lagrangian coordinates are ordered into an n-dimensional array $\mathbf{x}$, then the equilibrium equations can be written as:

$$\mathbf{K}\mathbf{x} = \mathbf{f}, \tag{1}$$

where $\mathbf{K}$ is the global stiffness matrix, and $\mathbf{f}$ is the array of the nodal forces.

The explicit formulae for the global stiffness matrix elements are given by Franciosi and Franciosi [4], together with a detailed analysis of the boundary conditions. It is only worth noting the peculiar highly banded structure of the matrix $\mathbf{K}$ (Fig. 2), which has been conveniently used to simplify the Gaussian solution routine.

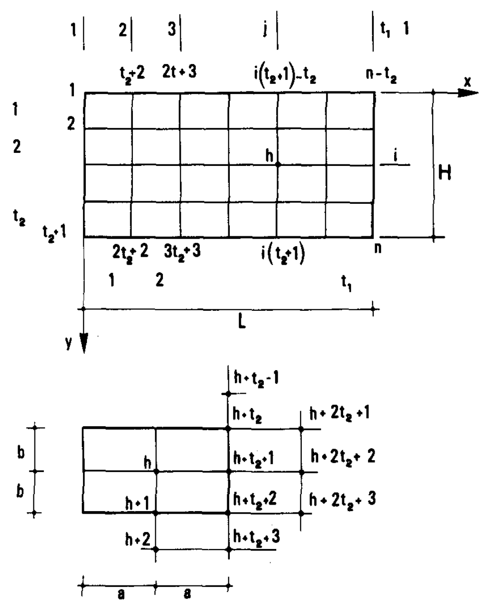

Fig. 1. Rectangular plate.

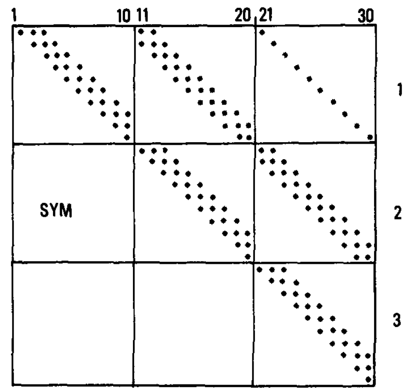

Fig. 2. Global stiffness matrix **K**.

3. NUMERICAL EXAMPLES

3.1. *Cantilever plates*

As a first example, consider a cantilever plate, subjected to the following three load conditions:

(a) two downward unit loads applied at the free edge (Fig. 3a),
(b) a distributed load along the free edge (Fig. 3b), and
(c) an upward unit force at one corner, and downward unit force at the other corner (Fig. 3c).

The first loading condition was examined in Sec. 2 by using a number of conforming and nonconforming

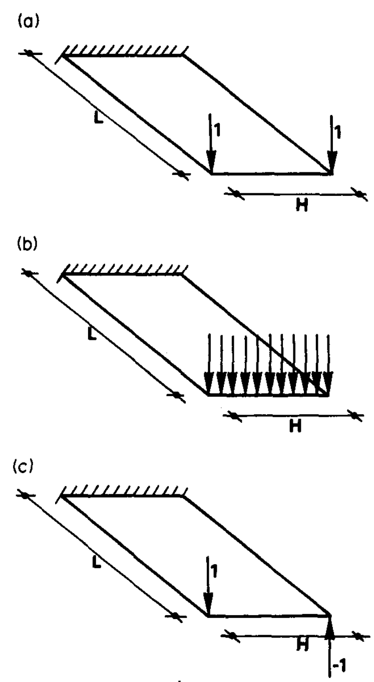

Fig. 3. Cantilever plate subjected to (a) two downward concentred unitary forces, (b) uniformly distributed load and (c) one downward unit force and one upward unit force.

Table 1. Results of cantilever plate—loading condition (a) (Fig. 3a): $E = 1000$, $\nu = 0.0$, $t = 0.1$, $L = 9.00$, $H = 3.00$

Element	d.o.f.	W_1	W_2	M_{max}
NC4 [2]	24	1919.07		−5.93
NC4 [2]	180	1940.95		−5.99
NC5 [2]	36	1939.77	1935	−5.99
NC5 [2]	288	1943.43		−6.00
NC6 [2]	42	1940.45	1934.8	−5.99
NC6 [2]	360	1943.32		−6.00
Beam theory		1944		−6.00

Table 2. Results of cantilever plate—loading condition (b) (Fig. 3a and b): $E = 1000$, $\nu = 0.0$, $t = 0.1$, $L = 9.0$, $H = 3.00$

		Load condition (a)		Load condition (b)	
Mesh	d.o.f.	W_{max}	M_{max}	W_{max}	M_{max}
3 × 3	16	2054.48	−6.00	2052.0	−6.00
9 × 3	40	1959.49	−6.00	1956.0	−6.00
18 × 6	133	1950.83	−6.00	1947.0	−6.00
27 × 9	280	1949.23	−6.00	1945.3	−6.00
36 × 12	481	1948.67	−6.00	1945.3	−6.00
54 × 18	1045	1948.27	−6.00	1944.3	−6.00
Beam theory				1944	−6.00

elements. The best results were achieved with nonconforming elements obtained by 2 × 2 Gaussian quadrature. In Table 1 some numerical results are reproduced from Choi and Park [2], where elements with four, five and six nodes are denoted by NC4, NC5 and NC6, respectively. In Table 2 the same quantities are reported, as obtained by the cell method.

The bending moment is always correctly calculated, and the greatest displacements for the load condition (a) are equal at the two corners. Obviously, the load condition (a) gives nonconstant displacements along the free edge, and consequently the cell method converges to a value somewhat higher than the value predicted by the beam theory. On the other hand, load condition (b) gives constant displacement along the free edge, and the method converges to the beam theory results. Load condition (c) is intended to check the behaviour of the cells method in the presence of twisting moment. Even this case was treated by Choi and Park [2], and again the best performance was offered by nonconforming five- and six-node plate bending elements obtained by 2 × 2 Gaussian quadrature.

In Table 3 the vertical displacements of the forces are given, for a number of different discretization

Table 3. Results of cantilever plate—loading condition (c) (Fig. 3c): $E = 1000$, $\nu = 0.3$, $t = 0.1$, $L = 9.00$, $H = 3.00$

Mesh	d.o.f.	W_1	W_2
NC5 [2]	36	−98.99	90.95
NC6 [2]	42	−95.56	87.75
[5]	180	−97.07	97.07
Eight-node element [2]	504	−98.76	98.76
3 × 3	16	−101.101	101.101
9 × 3	40	−97.63	97.63
18 × 6	133	−97.16	97.16
27 × 9	280	−96.988	96.988
36 × 12	481	−96.91	96.91

Table 8. Simply supported square plate (Fig. 5a and b) ($\nu = 0.3$). $\alpha_1 = DW_{max}/qL^4$, $\beta_1 = M_{max}/qL^2$, $\alpha_2 = DW_{max}/PL^2$

Mesh	d.o.f.	Load condition (a) α_1	Load condition (a) β_1	Load condition (b) α_2
[8]	27	0.003446		0.0013784
[8]	75	0.003939		0.0012327
[8]	243	0.004033		0.0011829
[8]	507	0.004050		0.0011715
[8]	867	0.004056		0.0011671
[9]	116	0.004057	0.0495563	0.00113847
[9]	280	0.0040623	0.0479467	0.00115486
[9]	516	0.0040623	0.0478986	0.00115776
[9]	824	0.0040621	0.0478876	0.00115873
2 × 2	9	0.00391	0.0423	0.001562
4 × 4	25	0.004028	0.0457	0.001367
10 × 10	121	0.004057	0.0462	0.0121
20 × 20	441	0.0040604	0.04779	0.011766
30 × 30	961	0.0040613	0.047844	0.001168
Exact [7]		0.0040623	0.0478864	0.0011604

The nondimensional central deflection coefficient for several meshes is reported in Table 8; again, it seems that the convergence rate of the proposed method is faster than the finite element convergence rate.

3.3. *Clamped square plate*

The nondimensional central deflection coefficient for a clamped square plate subjected to two different load conditions is reported in Table 9 (Fig. 6a and b).

This case seems to be the most difficult one to be treated by the cell method. In fact, it is necessary to use quite a refined mesh, if highly accurate results are required.

3.4. *Square tapered plate*

As a final example, consider the square simply supported plate with variable thickness in Fig. 7. The plate is subjected to an uniform load q, and its flexural rigidity D varies only in the vertical direction, according to the law:

$$D = D_0 e^{\lambda y} \qquad (2)$$

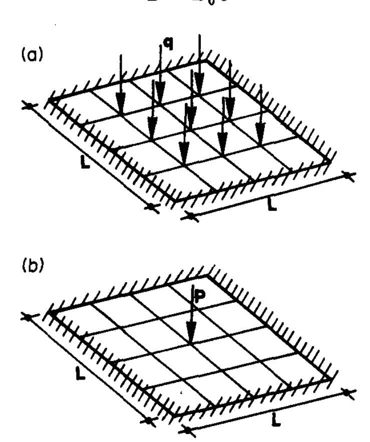

Fig. 6. Clamped square plate subjected to (a) uniformly distributed load and (b) unit force at the centre of the plate.

Table 9. Clamped square plate (Fig. 6a and b) ($\nu = 0.3$). $\alpha_1 = DW_{max}/qL^4$, $\alpha_2 = DW_{max}/PL^2$

Mesh	d.o.f.	Load condition (a) α_1	Load condition (b) α_1
[8]	27	0.001480	0.005919
[8]	75	0.001403	0.006134
[8]	243	0.001304	0.005803
[8]	507	0.001283	0.005710
[8]	867	0.001275	0.005672
4 × 4	25	0.00180	0.008075
10 × 10	121	0.001369	0.00623
20 × 20	441	0.001292	0.00580
30 × 30	961	0.001276	0.005706
Exact [7]		0.00126	0.00560

It is $D = D_0$ along the edge AD, and $D = 8D_0$ along the edge BC. This example has been studied in [3] and [10] by using 18 degrees of freedom triangular plate bending elements.

The numerical comparisons are shown in Table 10, in which the vertical displacements at six points are calculated with three different meshes.

4. CONCLUSIONS

A recently proposed discretization method has been applied to some classical test problem in plate bending analysis. The results have been compared

Table 10. Deflections/$(qL^2/10^4D_0)$ for the tapered plate in Fig. 7, $\nu = 0.25$

Point	Mesh 1 4 × 4	Mesh 2 8 × 8	Mesh 3 16 × 16	[2]	[10]
1	5.80	5.896	5.915	5.90	5.92
2	7.99	8.151	8.185	8.18	8.20
3	10.05	10.127	10.135	10.11	10.14
4	13.81	13.971	13.99	13.97	14.01
5	9.48	9.377	9.33	9.30	9.31
6	12.93	12.82	12.77	12.72	12.75

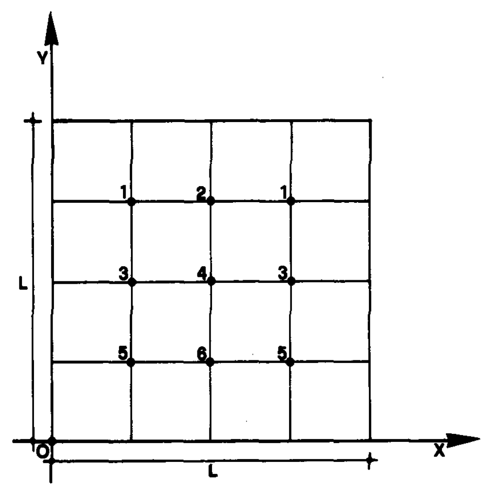

Fig. 7. Simply supported square plate with varying flexural rigidity along the y-direction.

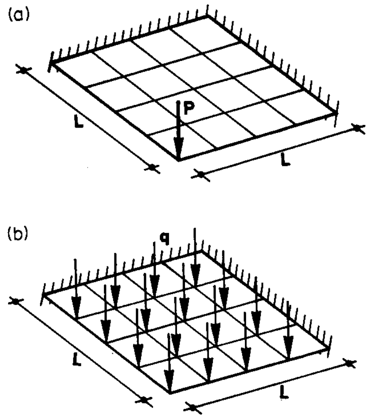

Fig. 4. Cantilever square plate with two clamped edges and two free edges, subjected to (a) a unit force at the free corner and (b) uniformly distributed load.

meshes. In this example, the convergence rate of the cells method compares favourably with the convergence rate of the finite element method.

As a second example, consider a plate with two clamped edges and two free edges. Two load conditions are examined:

(a) a concentrated load P at the free corner (Fig. 4a), and
(b) an uniformly distributed load q (Fig. 4b).

In the first case, Table 4 shows the greatest displacement and the greatest bending moment, according to the cell method and to three different finite element discretizations. It is evident that 176 degrees of freedom (d.o.f.) were not sufficient to achieve good accuracy in the finite elements methods. In fact, the same plate was studied [6] in an 'exact' way, by assuming $\nu = 0$; in Table 5 the greatest displacement is reported, for various discretization levels, and convergence to the true result is quite evident.

Table 4. Square plate with two clamped edges and two free edges—loading condition (a) (Fig. 4a): $E = 3600$, $\nu = 0.3$, $L = 10$, $t = 0.4$, $P = 100$

Mesh	d.o.f.	W_{max}	M_{max}
[2]	174	141.13	−110.0
[2]	176	143.12	−110.5
[2]	176	140.20	−170.0
4 × 4	25	142.47	−97.884
10 × 10	121	139.756	−103.516
20 × 20	441	139.18	−112.75
30 × 30	961	139.07	−114.073

Table 5. Square plate. Same data as Table 4, except that $\nu = 0$

Mesh	d.o.f.	W_{max}
4 × 4	25	127.79
10 × 10	121	126.053
20 × 20	441	125.64
30 × 30	961	125.55
Exact [6]		125.52

Table 6. Square plate with two clamped edges and two free edges—loading condition (b) (Fig. 4b): $E = 3600$, $\nu = 0.3$, $t = 0.4$, $L = 10$, $q = 0.9$

Mesh	d.o.f.	W_{max}	M_{max}
[2]	174	18.864	−24.60
[2]	176	19.066	−24.75
[2]	176	18.848	−24.60
4 × 4	25	20.44	−23.24
10 × 10	121	18.94	−24.71
20 × 20	441	18.68	−26.25
30 × 30	961	18.64	−26.37

Table 7. Square plate with two clamped edges and two free edges—loading condition (b) (Fig. 4b): $E = 1000$, $\nu = 0$, $L = 1$, $t = 0.1$, $q = 1$

Mesh	d.o.f.	W_{max}
4 × 4	25	0.4743
10 × 10	121	0.44158
20 × 20	441	0.4362
30 × 30	961	0.4351
Exact [6]		0.43404

The same plate, subjected to the load condition (b), was studied by three different finite element methods, and the numerical comparisons are given in Table 6. In this case, 176 degrees of freedom were enough to achieve satisfying results. The same case, for $\nu = 0$, is reported in Table 7, and again the convergence rate was quite rapid.

3.2. *Simply supported square plate*

This is a classical test problem, which has been extensively studied by means of different finite elements, and it was also solved by Timoshenko [7]. Two different load conditions will be considered:

(a) a concentrated force P at the centre of the plate (Fig. 5a) and
(b) an uniformly distributed load q (Fig. 5b).

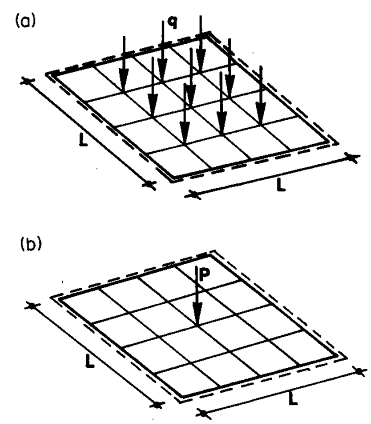

Fig. 5. Simply supported square plate subjected to (a) uniformly distributed load and (b) a unit force at the centre of the plate.

with various finite element outputs and—where available—with exact analytical results.

The convergence rate is shown to be quite rapid, and usually a coarse mesh will give satisfactory results. On the other hand, the peculiar banded form of the global stiffness matrix allows a considerable time saving, so that the proposed method seems to be at least as convenient as the most advanced finite element methods.

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