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Oscillations of Torus and Collision Torus-Chaos in a Delayed Circle Map

Claudio FRANCIOSI
Istituto di Scienza delle Costruzioni
Facoltà di Ingegneria, Università di Napoli
80125 Napoli
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A delayed version of the well-known circle map is examined, and a particular interesting scenario is closely followed. Oscillations of tori can be observed, according to an already explained mechanism, and a global bifurcation with hysteresis is illustrated, in which a strange attractor is suddenly destroyed by colliding with a coexisting saddle object. First numerical comparisons in the parameter plane with the circle map are also given.

The so-called circle map

$$\theta_{i+1}=\theta_i+A\ \sin(2\pi\theta_i)+D \tag{1}$$

is one of the most famous applications, because it naturally arises in the study of many different fields of physics and engineering.[1),2)] It has been extensively studied, from the classical study in the invertibility region $A<1/2\pi$[3),4)] until the recent works that establish some universal scale in both the region $A\leq 1/2\pi$[5)~8)] and $A>1/2\pi$.[9),10)] Quite recently, other papers have emphasized in the two-parameter context all the complexity of the noninvertibility region.[11)]

On the other hand, the following delayed version of the circle map

$$\theta_{i+1}=\theta_i+A\ \sin(2\pi\varphi_i)+D\,,$$
$$\varphi_{i+1}=\theta_i \tag{2}$$

is much less studied. It has some characteristic features of the circle map: for example, its parameter plane (A-D) has a symmetry line $D=.5$ and various Arnold horns can be detected in this plane.

However, it is well known that in a two-dimensional map another local bifurcation can be typically observed, i.e., the Neimark bifurcation,[12),13)] in which a periodic orbit of period n loses its stability and gives rise to n stable invariant circle surrounding the unstable n-periodic orbit. We shall see that this bifurcation is indeed present, and it is responsible for a global bifurcation with hysteresis.

In the following we will fix $A=.3$, while D will be allowed to increase from .5, but the chosen route is by no means pathological, and can be observed, with minor variations, for a wide range of A values. Increasing D from .5 we first encounter the biggest Arnold horn, in which a two-periodic orbit is stable. This orbit is born through saddle-node bifurcation at $D\cong.566$ and remains stable till $D\cong.606$, where it is destroyed through another saddle-node bifurcation. The resulting intermittency phenomenon can be observed on both the sides of the horn. The 3-periodic horn extends from $D\cong.75$ to $D\cong.763$, and a periodic orbit of period six can be observed in the range $.74<D<.742$, according to the general theory of Arnold horns.[3)] When D is increased past .742, the attractor becomes a torus that oscillates more and more strongly near the "future" 3-periodic orbit. (Fig. 1) These oscillations closely follow the unstable manifold of this orbit, as explained by Kaneko.[14)]

If the parameter is further increased, the attractor becomes strange and passes through

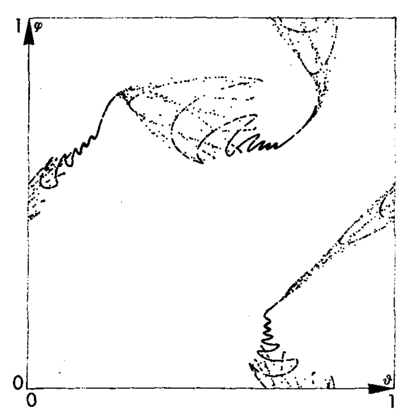

Fig. 1. Oscillations of the attractor near the 3-periodic Arnold horn.

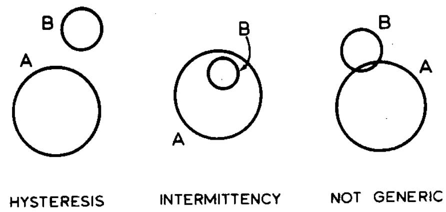

Fig. 2. Generic global bifurcations under a single control.

various other frequency lockings regions. Then for $D=.82$ it is suddenly replaced by a torus. This torus is located outside the portion of space previously occupied by the strange attractor, so we can say that a blue sky catastrophe has occurred.

In fact, it is known[15] that the only two generic global bifurcations that can be observed under the influence of a single parameter are intermittency, or interior crisis, and blue sky catastrophe, or boundary crisis. Both these phenomena are related to collision with saddle-type objects, but they are two totally different bifurcations, whose main characteristic can be summarized as in Fig. 2. As we can see, intermittency implies the passage from an attractor A to an attractor B which must be located inside A, or must inglobe A. On the other hand, the attractor B which is created by a blue sky catastrophe must be totally disjont from the attractor A. More than that, in this disjont case, the attractor B is not created at the moment of the bifurcation, but it must have already existed.[15] In our case, the strange attractor collides with a saddle object, and then it is replaced by the already existing stable torus. This bifurcation is usually associated with hysteresis, and indeed, if we let D decrease from $D=.82$ we observe that the torus shrinks and becomes a point attractor, through inverse Neimark bifurcation. This equilibrium point is destroyed at $D=.7$ (Fig. 3). If we now follow the upper path for increasing D values, we observe that the strange attractor and the saddle move and collide at the value $D=.82$. After that, the system jumps again to the torus attractor. We note that the saddle must lie between the stable torus and the strange attractor (Fig. 4), but it is useless to accurately detect its location, because the above qualitative arguments allow us to conclude that a blue sky catastrophe has occurred at a D value near to .82.

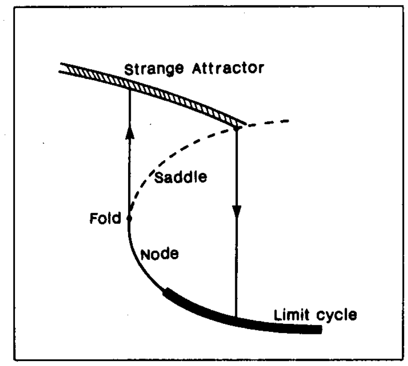

Fig. 3. Schematic diagram of the hysteresis associated to blue sky catastrophe.

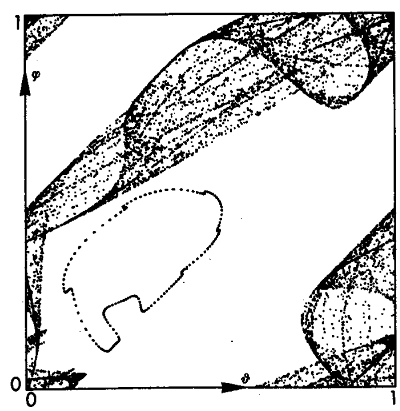

Fig. 4. Coexistence of torus and strange attractor for $A=.3$ and $D=.82$.

In this letter we have sketched some results about the behaviour of a delayed version of the circle map, emphasizing some global dynamical aspects that are not present in the circle map, and that are by no means pathological. Oscillations of torus near the 3-periodic frequency locking were illustrated, and a blue sky catastrophe was detected, with the resulting hysteresis.

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