Pergamon

0045-7949(95)00199-9

Computers & Structures Vol. 58, No. 6, pp. 1145–1159, 1996

0045-7949/96 $15.00 + 0.00

ON THE DYNAMIC BEHAVIOUR OF SLENDER BEAMS WITH ELASTIC ENDS CARRYING A CONCENTRATED MASS

M. A. De Rosa, C. Franciosi and **M. J. Maurizi**

Department of Structural Engineering and Soil Mechanics, University of Basilicata, Via della Tecnica 3, 85100 Potenza, Italy

(*Received* 2 *August* 1994)

Abstract—The dynamic behaviour of a slender beam carrying a concentrated mass at an arbitrary abscissa is examined. The beam is supposed to be elastically restrained against the rotation and the translation at both the ends, so that it is possible to study all the common boundary conditions. First, the exact solution is calculated, by solving the differential equations of motion and by imposing the corresponding boundary conditions. The resulting frequency equation is numerically solved. Subsequently, various approximate results are given, using the optimized Rayleigh–Schmidt approach with trigonometric and static shape functions, so that some comparison becomes possible. Finally, an application of the Morrow method allows us to obtain a lower bound to the true results.

INTRODUCTION

The free vibrations of slender beams carrying a concentrated mass is a classical argument of the structural dynamics, and numerous frequency equations have been deduced in the past. The prototype example is the cantilever beam carrying a concentrated mass at the tip, which can well exemplify a tower-tank structure, and both ideal clamp and elastically restrained ends have been dealt with [1, 2]. More recently, simply supported and clamped beams with mass at the midspan have been analyzed [3–5].

The Rayleigh method was also introduced in 1964 [6, 7], in order to obtain an upper bound to the fundamental frequency, and to this end both static and trigonometric trial functions have been extensively tested, for all the more common boundary conditions [3–5]. If non-classical boundary conditions are introduced, the exact analysis becomes more complicated, and Laplace transforms seem to be useful [8, 9]. In the presence of many intermediate masses and constraints, a computationally efficient approach can be based on the transfer matrix method [10].

Finally, the Timoshenko beam theory has also been used, in order to investigate the influence of the shear deformation and of the rotatory inertia on the dynamic behaviour of the beam-mass system [11, 12].

In this paper the beam-mass system of span l in Fig. 1 is examined, in which the mass M is placed at an arbitrary abscissa a from the left end, and both the ends are supposed to be elastically restrained against the rotation and against the translation. In this way, the constraints are defined by the four stifiness constants $k_{R1}, k_{T1}, k_{R2}, k_{T2}$, and all the classical boundary conditions can be recovered for limiting values of these stiffness coefficients.

First, the exact frequency analysis is performed, the differential equations are solved, the solutions are normalized, and the frequency equation is written down. The simply supported case is deduced, as a limiting case of the general frequency equation.

The Rayleigh quotient is then calculated, by using as trial function the static deflection of the beam with concentrated load at the mass abscissa. The final formulae are startlingly complicated, and are given in the Appendix.

As the next step, a one-parameter Rayleigh–Schmidt method is illustrated, mainly with reference to the simply supported beam, and a number of trial functions are used. It is shown that the trigonometric functions work slightly better than the static functions. The multiple-parameter Rayleigh–Schmidt is also considered and a convergence curve is reported for the clamped–clamped beam.

Finally, a complementary approach to the problem is briefly reviewed, and an effective lower bound is obtained, which works excellently even if the mass is placed near the ends of the beam.

The paper ends with some numerical comparisons, which hopefully can shed some light on the performance of the Rayleigh–Schmidt approach.

THE FREQUENCY EQUATION

It is well-known that the transverse vibrations of a slender beam can be described by the following differential equation of motion:

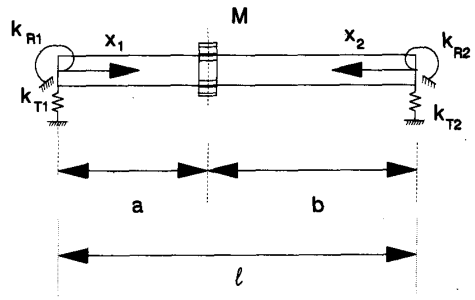

Fig. 1. The model.

$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+m\frac{\partial^2 w(x,t)}{\partial t^2}=0 \tag{1}$$

where E is the Young modulus, I is the second moment of area, and m is the distributed mass. Equation (1) can be written as

$$EI\frac{d^4 v(x)}{dx^4}-m\omega^2 v(x)=0 \tag{2}$$

if $w(x,t)$ is defined as

$$w(x,t)=v(x)\cos\omega t. \tag{3}$$

The solution of this differential equation is easily written down

$$v_1(x_1)=C_1\cosh kx_1+C_2\sinh kx_1 + C_3\cos kx_1+C_4\sin kx_1 \tag{4}$$

for $x_1\in[0,a]$, and

$$v_2(x_2)=C_5\cosh kx_2+C_6\sinh kx_2 + C_7\cos kx_2+C_8\sin kx_2 \tag{5}$$

for $x_2\in[0,b]$, and

$$k=\sqrt[4]{\frac{m\omega^2}{EI}}.$$

The boundary conditions are given by:

$$k_{R1}v_1'(x_1=0)=EIv_1''(x_1=0) \tag{6}$$

$$-k_{T1}v_1(x_1=0)=EIv_1'''(x_1=0) \tag{7}$$

$$k_{R2}v_2'(x_2=0)=EIv_2''(x_2=0) \tag{8}$$

$$-k_{T2}v_2(x_2=0)=EIv_2'''(x_2=0) \tag{9}$$

$$v_1(x_1=a)=v_2(x_2=b) \tag{10}$$

$$v_1'(x_1=a)=-v_2'(x_2=b) \tag{11}$$

$$v_1''(x_1=a)=v_2''(x_2=b) \tag{12}$$

$$EI[v_1'''(x_1=a)+v_2'''(x_2=b)]=-M\omega^2 v_1(x_1=a), \tag{13}$$

where the choice of the two different reference frames (Fig. 1) greatly simplifies the subsequent analysis.

It is also convenient to normalize the solutions, so that

$$v_i(x_i)=\sum_{j=1}^{4}C_{ij}V_{ij}(x_i)\quad i=1,2, \tag{14}$$

where C_{ij} are new integration constants, and

$$V_{i1}=\tfrac{1}{2}(\cosh kx_i+\cos kx_i) \tag{15}$$

$$V_{i2}=\frac{1}{2k}(\sinh kx_i+\sin kx_i) \tag{16}$$

$$V_{i3}=\frac{1}{2k^2}(\cosh kx_i-\cos kx_i) \tag{17}$$

$$V_{i4}=\frac{1}{2k^3}(\sinh kx_i-\sin kx_i). \tag{18}$$

After some algebra, the boundary conditions can be written as:

$$\mathbf{AC}=0, \tag{19}$$

where

$$\mathbf{C}^{\mathrm{T}}=\{C_{11},C_{12},C_{13},C_{14},C_{21},C_{22},C_{23},C_{24}\} \tag{20}$$

$$a_{11}=R_1aV_{12}(a)+V_{13}(a)$$

$$a_{12}=-T_1a^3V_{11}(a)+V_{14}(a) \tag{21}$$

$$a_{13}=-[R_2bV_{22}(b)+V_{23}(b)]$$

$$a_{14}=-[-T_2b^3V_{21}(b)+V_{24}(b)] \tag{22}$$

$$a_{21}=R_1aV_{12}'(a)+V_{13}'(a)$$

$$a_{22}=-T_1a^3V_{11}'(a)+V_{14}'(a) \tag{23}$$

$$a_{23}=[R_2bV_{22}'(b)+V_{23}'(b)]$$

$$a_{24}=[-T_2b^3V_{21}'(b)+V_{24}'(b)] \tag{24}$$

$$a_{31}=R_1aV_{12}''(a)+V_{13}''(a)$$

$$a_{32}=-T_1a^3V_{11}''(a)+V_{14}''(a) \tag{25}$$

$$a_{33}=-[R_2bV_{22}''(b)+V_{23}''(b)]$$

$$a_{34}=-[T_2b^3V_{21}''(b)+V_{24}''(b)] \tag{26}$$

$$a_{41} = R_1 a V'''_{12}(a) + V'''_{13}(a) + \frac{M\omega^2}{EI} a_{11}$$

$$a_{42} = -T_1 a^3 V'''_{11}(a) + V'''_{14}(a) + \frac{M\omega^2}{EI} a_{12} \quad (27)$$

$$a_{43} = [R_2 b V'''_{22}(b) + V'''_{23}(b)]$$

$$a_{44} = [-T_2 b^3 V'''_{21}(b) + V'''_{24}(b)] \quad (28)$$

and

$$R_1 = \frac{EI}{K_{R1} a} \quad R_2 = \frac{EI}{K_{R2} b} \quad T_1 = \frac{EI}{K_{T1} a^3} \quad T_2 = \frac{EI}{K_{T2} b^3}. \quad (29)$$

The frequency equation

$$\det(\mathbf{A}) = 0 \quad (30)$$

can be numerically solved by means of the suggestions reported in Ref. [13]. Actually, all the numerical results which are given later were found by employing a trivial bisection routine, and no troubles occurred, probably because the eigenvalues were always well scattered along the real axis.

On the other hand, some degenerate cases can be worked out and the frequency equation can be written down explicitly. As an example, the simply supported beam can be studied by putting $T_1 = T_2 = 0$, and $R_1 = R_2 = \infty$. After some algebra we have the frequency equation

$$Mk\{\cosh[k(a-b)]\sin(kl) - \cosh(kl)\sin(kl) + \cos[k(a-b)]\sinh(kl) - \cos(kl)\sinh(kl)\} - 4m \sin(kl)\sinh(kl) = 0. \quad (31)$$

THE RAYLEIGH APPROACH TO THE UPPER BOUNDS

A useful upper bound can be obtained by approximating the first frequency by means of the Rayleigh quotient

$$\omega^2 = \frac{U_{max}}{T_{max}} \quad (32)$$

where U_{max} is the maximum total potential energy and T_{max} is the maximum kinetic energy.

It is

$$U_{max} = \frac{EI}{2}\int_0^l v''(x)^2\,dx + \tfrac{1}{2}k_{R1}\phi_1^2 + \tfrac{1}{2}k_{R2}\phi_2^2 + \tfrac{1}{2}k_{T1}v_1^2 + \tfrac{1}{2}k_{T2}v_2^2 \quad (33)$$

and

$$T_{max} = \frac{m}{2}\int_0^l v(x)^2\,dx + \tfrac{1}{2}Mv^2(a), \quad (34)$$

where v_1, v_2, ϕ_1 and ϕ_2 are the vertical displacements and the rotations at left and right ends, respectively.

It is evident that the choice of the approximating function is of paramount importance, in order to obtain a close upper bound. A common procedure, which goes back to Timoshenko [7], consists of calculating the static deflection of the beam with concentrated load at the mass abscissa. This deflection is subsequently introduced in eqns (33) and (34). The whole procedure becomes quite cumbersome, because of the flexibile ends and the Rayleigh quotient is rather elaborate. Anyhow, the whole expression is given in the Appendix.

In some instances, it could be important to use some other *ad hoc* trial function, which is strictly related to the particular boundary conditions. As an example, let us consider the simply supported beam, for which $k_{R1} = k_{R2} = 0$, $v_1 = v_2 = 0$. Three different classes of shape functions can be used:

(1) the simplest polynomial function: $v^{(1)} = x(l-x)$;
(2) the static deflection $v^{(2)}$ of the simply supported beam with concentrated load at $x = a$;
(3) the trigonometric function

$$v^{(3)} = \sin\frac{\pi x}{l} \quad (35)$$

resulting in the following approximations

$$\bar{\omega}^2 = \frac{4EIl}{ml^5/30 + a^2b^2M} \quad (36)$$

$$\bar{\omega}^2 = \frac{315EIl}{m(3a^4l - 6a^3l^2 - a^2l^3 + 4al^4 + 2l^5) + 105Ma^2(a-l)^2} \quad (37)$$

$$\bar{\omega}^2 = \frac{EI\pi^4}{ml^4 + M/2\sin(a\pi/l)^2} \quad (38)$$

respectively. It is well known that the trigonometric functions should be preferred, at least if the mass is placed near the ends of the beam [3–5].

THE RAYLEIGH–SCHMIDT APPROACH TO THE UPPER BOUNDS

If a closer approximation is needed, it is possible to employ a modified Rayleigh–Schmidt procedure [14, 15]. According to it, a shape function is selected, which depends on an unknown multiplier

$$v = v(x, \mu). \quad (39)$$

It follows that the resulting approximate frequency also depends on the same multiplier

$$\bar{\omega}^2 = \bar{\omega}^2(\mu) \tag{40}$$

and can be minimized with respect to it. The condition

$$\frac{d\bar{\omega}^2}{d\mu} = 0 \tag{41}$$

allows us to obtain the unknown multiplier μ.

In the following, a number of trial functions have been considered, because the use of an already developed MATHEMATICA [16] package [17] greatly simplifies the analysis. For the simply supported beam, the following one-parameter shape functions have been adopted:

(1) $v^{(1)} \times (1 + \mu v^{(1)})$
(2) $v^{(2)} \times (1 + \mu v^{(2)})$
(3) $\sin(\pi x)/l + \mu \sin(2\pi x)/l$
(4) $\sin(\pi x)/l \times (1 + \mu v^{(1)})$.

Usually, the Rayleigh–Schmidt optimization involves large computational efforts, and the final result is often unmanageable. As an example, let us consider in detail the trigonometric choice, which leads to the quotient

THE MULTIPLE PARAMETER RAYLEIGH–SCHMIDT APPROACH TO THE UPPER BOUNDS

As already pointed out in Ref. [17], the MATHEMATICA package allows us to easily obtain a higher approximation by using a two-parameter Rayleigh–Schmidt procedure. For the simply supported beam, a large sample of trial functions can be deduced by combining the three basic above-mentioned functions, as for example:

(1) $v^{(1)} \times (1 + \mu v^{(1)} + \mu_1 v^{(1)2})$
(2) $v^{(2)} \times (1 + \mu v^{(2)} + \mu_1 v^{(2)2})$
(3) $\sin(\pi x/l) + \mu \sin(3\pi x/l) + \mu_1 \sin(5\pi x/l)$
(4) $\sin(\pi x/l) + \mu \sin(3\pi x/l) + \mu_1 v^{(1)}$
(5) $v^{(2)} \times (1 + \mu v^{(2)} + \mu_1 v^{(1)2})$.

Nonetheless, the formulae become unmanageable, and a numerical approach must be used in order to solve the nonlinear optimization conditions. Moreover, it is known that the improvements of the results are usually too small to justify the intricacy of the analysis. In the following, we shall show that some interesting improvement can be obtained by blending trigonometric and static trial functions.

$$\bar{\omega}^2(\mu) = \frac{\pi^4 EI(1 + 16\mu^2)}{l^3(ml(1 + \mu^2) + 2M(\sin(a\pi/l) + \mu \sin(2a\pi/l))^2)}. \tag{42}$$

The minimum condition results in the equation

$$\left(32M \sin\frac{\pi a}{l} \sin\frac{2\pi a}{l}\right)\mu^2 + \left(15ml + 32\sin^2\frac{\pi a}{l} - 2M\sin^2\frac{2\pi a}{l}\right)\mu + 2M \sin\frac{\pi a}{l}\sin\frac{2\pi a}{l} = 0 \tag{43}$$

with solutions

$$\mu_{1,2} = \frac{1}{64M}\left\{\csc\frac{a\pi}{l}\csc\frac{2a\pi}{l}\left[-15ml - 32M\sin^2\frac{a\pi}{l} + 2M\sin^2\frac{2a\pi}{l} \mp \sqrt{256M^2\sin^2\frac{a\pi}{l}\sin^2\frac{2a\pi}{l} + \left(15ml + 32M\sin^2\frac{a\pi}{l} - 2M\sin^2\frac{2a\pi}{l}\right)^2}\right]\right\}. \tag{44}$$

The optimum Rayleigh quotient can be obtained by inserting these roots into eqn (42). It will be

$$\bar{\omega}^2_{1,2} = \frac{16EI\pi^4}{l^3}\frac{256M^2 + \csc^2\frac{a\pi}{l}\csc^2\frac{2a\pi}{l}A^2}{2M\left(\sin\frac{\pi a}{l} + A\csc\frac{\pi a}{l}\right)^2 + ml\left(4096M^2 + \csc^2\frac{a\pi}{l}\csc^2\frac{2\pi a}{l}A^2\right)}$$

$$A = -15ml - 32M\sin^2\frac{\pi a}{l} + 2M\sin^2\frac{2\pi a}{l} \pm \sqrt{R}$$

$$R = 256M\sin^2\frac{\pi a}{l}\sin^2\frac{2\pi a}{l} + \left(15ml + 32M\sin^2\frac{\pi a}{l} - 2M\sin^2\frac{2\pi a}{l}\right)^2.$$

THE MORROW APPROACH TO THE LOWER BOUNDS

Let us consider the beam in Fig. 1. The bending moment at the generic abscissa x will be given by the following integro-differential equation:

$$M(x) = M_A - m \int_0^x (x - x') \frac{\partial^2 w(x, t)}{\partial t^2} \mathrm{d}x' - M\langle x - a\rangle^0 (x - a) \frac{\partial^2 w(a, t)}{\partial t^2}, \quad (45)$$

where M_A is the bending moment due to the left constraint, and $\langle\rangle$ are the McCauley brackets. Basically, the Morrow method solves this equation in an iterative way, which will briefly be reviewed here.

As a preliminary step, let us use eqn (3), in order to write

$$\frac{\ddot{w}(x, t)}{w(x, t)} = \frac{\ddot{w}(a, t)}{w(a, t)} = -\omega^2. \quad (46)$$

This relationship is inserted into eqn (45), and we have

$$-EI \frac{\partial^2 w(x, t)}{\partial x^2} = M_A - m \frac{\partial^2 w(a, t)}{\partial t^2} \int_0^x (x - x') \times \frac{w(x, t)}{w(a, t)} \mathrm{d}x' - M\langle x - a\rangle^0 (x - a) \frac{\partial^2 w(a, t)}{\partial t^2}. \quad (47)$$

In order to start the iterative routine, we put

$$\frac{w(x, t)}{w(a, t)} = \frac{w^{(0)}(x, t)}{w^{(0)}(a, t)}, \quad (48)$$

where $w^{(0)}(x, t)$ must satisfy the boundary conditions. For example, it can be $w^{(0)}(x, t) = v^{(2)} \cos \omega t$. Equation (46) can now be integrated, and the solution can be written as

$$w(x, t) = u^{(1)}(x) \frac{\partial^2 w(a, t)}{\partial t^2}. \quad (49)$$

Table 1. Simply supported beam—(1) lower bound to the first nondimensional frequency [19], (2) exact result, (3) upper bound by the Rayleigh–Schmidt method

			a/l		
M/m	0.1	0.2	0.3	0.4	0.5
0.1	95.56	90.98	85.91	82.03	79.80
	95.57018	91.03107	86.01879	82.41856	81.1408
	95.5726	91.0338	86.0255	82.4288	81.1461
0.2	93.74	85.20	76.71	70.85	67.73
	93.77472	85.30814	76.86739	71.35174	69.49377
	93.78338	85.31735	76.88684	71.37858	69.50707
0.3	91.95	80.02	69.22	62.35	58.99
	92.02226	80.16679	69.3912	62.86802	60.75401
	92.04089	80.18430	69.4237	62.90950	60.77404
0.4	90.21	75.36	63.01	55.68	52.31
	90.31271	75.53871	63.18898	56.16739	53.95822
	90.34439	75.56514	63.23300	56.22011	53.98320
0.5	88.69	71.17	57.81	50.30	47.03
	88.64575	71.36222	57.97207	50.74603	48.52488
	88.69309	71.39747	58.02550	50.80691	48.55329
0.6	86.86	67.39	53.91	45.86	42.74
	87.02095	67.58263	53.52971	46.27213	44.08272
	87.08613	67.62619	53.59056	46.33870	44.11341
0.7	85.24	63.96	49.56	42.15	39.18
	85.43779	64.15203	49.70545	42.51875	40.38381
	85.52258	64.20320	49.77201	42.58913	40.41594
0.8	83.67	60.84	46.25	38.99	36.17
	83.89562	61.02871	46.38132	39.32567	37.25638
	84.00149	61.08669	46.45216	39.39848	37.28936
0.9	82.15	57.99	43.35	36.27	33.60
	82.39378	58.1765	43.46694	36.5757	34.57771
	82.52186	58.24048	43.54091	36.65091	34.61109
1	80.66	55.39	40.78	33.91	31.38
	80.93149	55.56409	40.8921	34.18555	32.25783
	81.08266	55.63329	40.96829	34.26040	32.29130

Table 2. Clamped beam—(1) lower bound to the first nondimensional frequency [19], (2) exact result, (3) upper bound by the Rayleigh–Schmidt method

M/m	a/l 0.1	0.2	0.3	0.4	0.5
0.1	499.9 498.76313 518.255	480.9 481.48415 503.489	443.9 445.19265 466.253	411.2 411.82814 426.472	399.0 399.18006 410.144
0.2	497.9 496.94327 517.001	460.8 462.75422 488.422	396.7 398.62996 422.896	347.9 348.58305 361.694	331.7 331.4361 338.814
0.3	495.9 495.10371 515.753	441.3 444.50538 474.23	357.4 359.5671 386.917	301.1 301.64471 314	283.0 283.13968 288.62
0.4	493.2 493.24455 514.511	422.6 426.84543 460.84	324.6 316.68032 356.58	265.2 265.58648 277.418	247.0 247.03032 251.378
0.5	491.8 491.36587 513.275	404.7 409.85618 448.185	296.9 298.81138 330.654	236.7 237.09051 248.471	219.0 219.03875 222.649
0.6	489.6 489.4678 512.045	387.8 393.59403 436.207	273.3 275.00873 308.243	213.8 214.03875 224.994	196.7 196.71686 199.814
0.7	487.3 487.55048 510.82	371.9 378.09224 424.852	253.0 254.51195 288.676	194.8 195.02547 205.571	178.5 178.50708 181.226
0.8	484.9 485.61408 509.602	357.0 363.36428 414.074	235.4 236.71956 271.446	178.9 179.08513 189.234	163.3 163.37255 165.803
0.9	482.6 483.65877 508.389	342.9 349.40757 403.829	217.8 221.15642 256.157	165.4 165.53449 175.303	150.6 150.59704 152.799
1	480.1 481.68477 507.182	329.8 336.20713 394.078	206.4 207.44583 242.498	153.7 153.8773 163.283	139.7 139.67014 141.686

It follows from eqn (45) that the frequency can be approximated as

$$(\omega_1^{(1)})^2 = -\frac{1}{u(a)} \tag{50}$$

and also

$$\frac{w(x,t)}{w(a,t)} = \frac{u^{(1)}(x)}{u^{(1)}(a)}. \tag{51}$$

The procedure can be repeated by assuming this ratio as new starting point.

Two points must be stressed here: first of all, the method is known to converge very rapidly to the correct result, and, even more important, the convergence can be proved to be from below. This is particularly important, because it allows us to obtain a lower–upper bound to the true frequency.

Finally, the primary reference to this method seems to be Ref. [18], but the authors were unable to read the original paper, and their source is Ref. [19].

NUMERICAL RESULTS

In Table 1 the first nondimensional frequency coefficient

$$\beta = \frac{ml^4}{EI}\omega^2 \tag{52}$$

is reported, for a simply supported beam, and for various mass ratios M/m and mass abscissae. For each entry of the table, we give the lower bound—as given in Ref. [19] by using the Morrow method—the exact result and the upper bound, as obtained by using the Rayleigh–Schmidt procedure with trial function

$$v(x) = \sin\frac{\pi x}{l} + \mu\sin\frac{2\pi x}{l} + \mu_1\sin\frac{3\pi x}{l}. \tag{53}$$

It is interesting to note that the Morrow method gives better results if the mass is placed near the ends. On the other hand, the upper bound is always very close to the true frequency.

In Table 2 the same results are given for a clamped–clamped beam. The classical Rayleigh method was applied with the trigonometric trial function

$$v(x) = 1 - \cos\frac{2\pi x}{l}. \qquad (54)$$

It is immediately seen that the Morrow method works excellently, whereas the upper bounds are often very imprecise. With this in mind, a convergence curve is reported in Table 3, by using the Rayleigh–Schmidt optimization approach, with trigonometric functions:

$$v(x) = 1 - \cos\frac{2\pi x}{l} + \mu\left(1 - \cos\frac{4\pi x}{l}\right) \qquad (55)$$

$$v(x) = 1 - \cos\frac{2\pi x}{l} + \mu\left(1 - \cos\frac{4\pi x}{l}\right) + \mu_1\left(1 - \cos\frac{6\pi x}{l}\right) \qquad (56)$$

$$v(x) = 1 - \cos\frac{2\pi x}{l} + \mu\left(1 - \cos\frac{4\pi x}{l}\right) + \mu_1\left(1 - \cos\frac{6\pi x}{l}\right) + \mu_2\left(1 - \cos\frac{8\pi x}{l}\right), \qquad (57)$$

Table 3. Clamped beam—approximate nondimensional frequency Rayleigh approach and Rayleigh–Schmidt approach with one, two and three multipliers

			a/l		
M/m	0.1	0.2	0.3	0.4	0.5
0.1	518.255	503.489	466.253	426.472	410.144
	503.451	486.756	450.316	416.319	402.941
	500.549	483.560	448.245	414.370	400.259
	499.617	482.698	447.507	413.524	399.821
0.2	517.001	488.422	422.896	361.694	338.814
	501.910	469.573	406.130	354.057	334.743
	498.875	465.972	404.532	352.585	332.234
	497.878	465.121	403.894	351.782	331.984
0.3	515.753	474.23	386.917	314	288.62
	500.375	453.381	369.727	307.960	286.101
	497.203	449.400	368.460	306.793	283.778
	496.140	448.560	367.900	306.039	283.623
0.4	514.511	460.84	356.58	277.418	251.378
	498.846	438.113	339.240	272.466	249.715
	495.535	433.781	338.214	271.508	247.566
	494.404	432.954	337.716	270.801	247.464
0.5	513.275	448.185	330.654	248.471	222.649
	497.322	423.709	313.351	244.299	221.494
	493.870	419.054	312.505	243.490	219.504
	492.670	418.241	312.056	242.828	219.433
0.6	512.045	436.207	308.243	224.994	199.814
	495.803	410.110	291.102	221.404	198.979
	492.210	405.160	290.391	220.707	197.130
	490.939	404.359	289.985	220.086	197.079
0.7	510.82	424.852	288.676	205.571	181.226
	494.291	397.260	271.780	202.429	180.604
	490.553	392.044	271.176	201.819	178.880
	489.209	391.257	270.805	201.234	178.841
0.8	509.602	414.074	271.446	189.234	165.803
	492.783	385.108	254.848	186.448	165.327
	488.900	379.653	254.328	185.906	163.712
	487.482	378.881	253.986	185.354	163.683
0.9	508.389	403.829	256.157	175.303	152.799
	491.282	373.606	239.890	172.803	152.426
	487.251	367.938	239.438	172.317	150.910
	485.757	367.179	239.122	171.795	150.887
1	507.182	394.078	242.498	163.283	141.868
	489.786	362.710	226.582	161.019	141.389
	485.605	356.852	226.185	160.579	139.961
	484.034	356.107	225.891	160.084	139.943

Table 4. Clamped–clamped beam—Rayleigh quotient with polynomial trial function

M/m	0.1	0.2	a/l 0.3	0.4	0.5
0.1	501.9532	484.0161	448.9886	416.8669	404.4639
0.2	499.8677	465.5565	404.8043	355.4207	337.7591
0.3	497.8268	448.4532	368.5371	309.7618	289.9416
0.4	495.8025	432.5621	338.2341	274.4985	253.9843
0.5	493.7947	417.7586	312.5358	246.4434	225.9615
0.6	491.803	403.9348	290.4668	223.5912	203.5079
0.7	489.8273	390.9966	271.3089	204.6175	185.1133
0.8	487.8675	378.8615	254.5218	188.6121	169.7684
0.9	485.9232	367.457	239.691	174.9289	156.7728
1	483.9944	356.719	226.4934	163.0968	145.6253

Table 5. Clamped–clamped beam—Rayleigh–Schmidt approach: (1) polynomial functions and (2) trigonometric and polynomial functions

M/m	0.1	0.2	a/l 0.3	0.4	0.5
0.1	498.9264	482.106	447.0857	413.131	399.3263
	498.7741	481.9381	446.8220	412.9811	399.3684
0.2	497.1023	464.5467	403.6703	351.5204	331.6815
	496.9857	464.4097	403.3982	351.4024	331.8867
0.3	495.2835	448.011	367.8579	305.8623	283.4774
	495.2000	447.9104	367.6050	305.7680	283.7818
0.4	493.4702	432.4338	337.8297	270.6819	247.4331
	493.4171	432.3727	337.6067	270.6053	247.7949
0.5	491.6622	417.7521	312.2995	242.7487	219.483
	491.6370	417.7322	312.1092	242.6856	219.8760
0.6	489.8598	403.9058	290.3335	220.0349	197.1853
	489.8598	403.9348	290.1750	219.9822	197.5934
0.7	488.063	390.8382	271.2381	201.2042	178.9878
	488.0858	390.9966	271.1092	201.1596	179.4009
0.8	486.2718	378.4959	254.4879	185.3399	163.8574
	486.3149	378.8615	254.3855	185.3017	164.2693
0.9	484.4863	366.8288	239.6776	171.7928	151.0806
	484.5473	367.4570	239.5988	171.7598	151.4875
1	482.7066	355.79	226.4901	160.09	140.1488
	482.7830	356.71900	226.4319	160.0613	140.5482

Table 6. Simply supported beam—approximate nondimensional first frequency. Rayleigh approach

M/m	a/l 0.1	0.2	0.3	0.4	0.5
0.1	117.153	111.441	105.979	102.319	101.053
	127.564	105.506	91.6937	84.2787	81.9512
	95.584	91.1133	86.134	82.487	81.1742
0.2	114.438	104.022	94.8917	89.1795	87.2727
	123.317	96.792	80.8024	72.5405	70
	93.8253	85.582	77.1983	71.5295	69.5779
0.3	111.846	97.5293	85.9045	79.0306	76.8
	119.343	89.4074	72.2237	63.6724	61.0909
	92.1305	80.6838	69.9424	63.1417	60.8807
0.4	109.369	91.7993	78.4724	70.9555	68.5714
	115.618	83.0696	65.2917	56.7363	54.1936
	90.4958	76.3159	63.9332	56.5147	54.1162
0.5	107	86.7052	72.2239	64.3777	61.9355
	112.118	77.5709	59.5739	51.1629	48.6957
	88.9182	72.3967	58.8749	51.1466	48.7045
0.6	104.73	82.1468	66.8971	58.9159	56.4706
	108.823	72.755	54.7769	46.5866	44.2105
	87.3946	68.8603	54.5584	46.7098	44.2769
0.7	102.555	78.0437	62.3021	54.3085	51.8919
	105.717	68.5021	50.6948	42.7617	40.4819
	85.9223	65.6533	50.8315	42.9813	40.5871
0.8	100.469	73.331	58.2977	50.3694	48
	102.783	64.7189	47.179	39.5173	37.3333
	84.4988	62.7318	47.5813	39.8041	37.465
0.9	98.4656	70.9555	57.777	46.9631	44.6512
	100.008	61.3318	44.1192	36.7304	34.6392
	83.1217	60.0592	44.7217	37.0642	34.789
1	96.5406	67.8733	51.6573	43.9883	41.7391
	97.3785	58.2815	41.4321	34.3107	32.3077
	81.7888	57.605	42.1864	34.6773	32.4697

respectively. Undoubtedly, an improvement of the results can be noticed, but—at least in some cases—the approximation remains quite bad.

If the simple trial function:

$$v(x) = x^2(l-x)^2 \tag{58}$$

is used, then the Rayleigh quotient can be written as

$$\bar{\omega}^2 = \frac{4EIl^5}{ml^9/126 + 5a^4b^4M} \tag{59}$$

and the first nondimensional frequency is estimated as in Table 4. A further improvement can be obtained with the Rayleigh–Schmidt approach and the trial function

$$v(x) = x^2(l-x)^2[1 + \mu(x^2(l-x)^2)]. \tag{60}$$

As already said, it seems more convenient to blend trigonometric and polynomial functions, by using the trial function:

$$v(x) = 1 - \cos\frac{2\pi x}{l} + \mu x^2(l-x)^2. \tag{61}$$

A glance at Table 5 will confirm this statement.

In Table 6, the nondimensional frequency for a simply supported beam is reported, as obtained by using eqns (36)–(38), respectively. Again, the simple quadratic trial function behaves surprisingly well. Finally, in Table 7 some Rayleigh–Schmidt approaches for the simply supported beam are reported.

The first entry refers to the trial function

$$v(x) = v^{(2)} \times (1 + \mu v^{(2)}) \tag{62}$$

and it is immediately seen that this choice leads to a poor frequency prediction if the mass is placed near the end, but also that it conducts to the best approximation, if the mass is placed at the midspan.

The second entry uses the trial function

$$v(x) = v^{(1)} \times (1 + \mu v^{(1)}) \tag{63}$$

Table 7. Simply supported beam, Rayleigh–Schmidt approach with one unknown multiplier: (1) static trial function, (2) quadratic trial function, (3) trigonometric trial function, (4) trigonometric and quadratic trial function

M/m	a/l 0.1	0.2	0.3	0.4	0.5
0.1	121.483	104.967	91.673	83.9712	81.497
	95.6219	91.1222	86.167	82.5646	81.2711
	95.5755	91.0465	86.0276	82.4381	81.1742
	95.5809	91.1071	86.1337	82.4821	81.1624
0.2	117.799	96.375	80.788	72.344	69.717
	93.8413	85.5463	77.2149	71.6111	69.6822
	93.7945	85.3600	76.8930	71.4028	69.5779
	93.8151	85.5615	77.1974	71.5167	69.5480
0.3	114.322	89.0786	72.2134	63.5398	60.9025
	92.1219	80.6084	69.9477	63.2215	60.9818
	92.0647	80.2653	69.4340	62.9468	60.8807
	92.1088	80.6451	69.9409	63.1220	60.8357
0.4	111.037	82.8059	65.2841	56.6425	54.062
	90.4609	76.2058	63.9308	56.5904	54.2107
	90.3848	75.6874	63.2470	56.2676	54.1161
	90.4592	76.2578	63.9312	56.4895	54.0600
0.5	107.927	77.3561	59.5681	51.0942	48.6002
	88.8556	72.2563	58.8669	51.2176	48.7917
	88.7533	71.5604	58.0424	50.8617	48.7045
	88.8638	72.3192	58.8725	51.1174	48.6406
0.6	104.98	72.5778	54.7724	46.5347	44.1391
	87.3033	68.6939	54.5464	46.7762	44.3569
	87.1689	67.8276	53.6099	46.3986	44.2769
	87.3202	68.7644	54.5556	46.6778	44.2078
0.7	102.185	68.3542	50.6913	42.7216	40.4271
	85.8015	65.4645	50.8167	43.0434	40.6607
	85.6301	64.4397	49.7931	42.6525	40.5871
	85.8261	65.5402	50.8285	42.9474	40.5147
0.8	99.5296	64.5942	47.1761	39.4856	37.2904
	84.3481	62.5238	47.5644	39.8621	37.5328
	84.1356	61.3548	46.4746	39.4640	37.4650
	84.3793	62.6030	47.5780	39.7689	37.3907
0.9	97.0044	61.2256	44.1168	36.705	34.6049
	82.9408	59.8349	44.7033	37.1186	34.8515
	82.6840	58.5364	43.5644	36.7177	34.7890
	82.9776	59.9162	44.7182	37.0283	34.7137
1	94.6004	58.1904	41.4302	34.29	32.2798
	81.5776	57.3668	42.1669	34.7283	32.5277
	81.2738	55.9535	40.9924	34.3278	32.4697
	81.6193	57.4493	42.1828	34.6410	32.3941

and it certainly gives better results than the first choice, when the mass is placed near the ends. The third entry refers to the trigonometric functions, and the fourth entry blends trigonometric and quadratic functions

$$v(x) = v^{(1)} \times \left(1 + \mu \sin \frac{\pi x}{l}\right). \qquad (64)$$

It is worth noting that it is not easy to deduce the "better" choice, as each trial function has its advantages and its drawbacks.

CONCLUSIONS

In this paper various approaches to the dynamic analysis of mass-beam systems with elastic ends have been presented. The exact solution has been achieved, in the presence of translational and rotational flexible constraints, so that every kind of boundary condition can be treated as a limiting case. The classical Rayleigh quotient has been calculated for the same structural system, and its optimized version has been used for simply supported and clamped beams. An iteration method—which goes back to Morrow [18]—allowed us to solve the integro–differential equation of the bending moment, so enabling the attainment of a lower bound to the true results.

Finally, some numerical examples lead one to deduce that the trigonometric functions are not always the right choice, but sometimes the simplest polynomial functions can give better results.

Acknowledgement—M. J. Maurizi wishes to acknowledge the support of Università degli Studi della Basilicata, which made this work possible. Work partially supported by Italian MURST.

REFERENCES

1. P. A. A. Laura, J. L. Pombo and E. A. Susemihl, A note on the vibrations of a clamped–free beam with a mass at the free end. *J. Sound Vibr.* **37,** 161–168 (1974).
2. P. A. A. Laura, M. J. Maurizi and J. L. Pombo, A note on the dynamic analysis of an elastically restrained-free beam with a mass at the free end. *J. Sound Vibr.* **41,** 397–405 (1975).
3. G. B. Chai and K. H. Low, On the natural frequencies of beams carrying a concentrated mass. *J. Sound. Vibr.* **160,** 161–166 (1993).
4. K. H. Low, T. M. Lim and G. B. Chai, Experimental and analytical investigations of vibration frequencies for centre-loaded beams. *Comput. Struct.* **48,** 1157–1162 (1993).
5. K. H. Low, An equivalent-center method for quick frequency analysis of beams carrying a concentrated mass. *Comput. Struct.* **50,** 409–419 (1994).
6. W. C. Hurty and M. F. Rubinstein, *Dynamics of Structures.* Prentice-Hall, Englewood Cliffs, NJ (1964).
7. S. Timoshenko, D. H. Young and W. Weawer jr., *Vibration Problems in Engineering,* 4th Edn. Van Nostrand, New York (1974).
8. R. P. Goel, Vibrations of a beam carrying a concentrated mass. *J. appl. Mech.* **40,** 821–822 (1973).
9. R. P. Goel, Free vibrations of a beam-mass system with elastically restrained ends. *J. Sound Vibr.* **47,** 9–14 (1976).
10. C. N. Bapat and C. Bapat, Natural frequencies of a beam with non-classical boundary conditions and concentrated masses. *J. Sound Vibr.* **112,** 177–182 (1987).
11. M. J. Maurizi and P. M. Bellés, Natural frequencies of the beam-mass system: comparison of the two fundamental theories of beam vibrations. *J. Sound Vibr.* **150,** 330–334 (1991).
12. M. J. Maurizi and P. M. Bellés, An additional evaluation of free vibrations of beam-mass systems. *J. Sound Vibr.* **154,** 182–186 (1992).
13. K. H. Low, A comprehensive approach for the eigenproblem of beams with arbitrary boundary conditions. *Comput. Struct.* **39,** 671–678 (1991).
14. C. W. Bert, Application of a version of the Rayleigh technique to problems of bars, beams, columns, membranes and plates. *J. Sound Vibr.* **119,** 317–326 (1987).
15. P. A. A. Laura and V. H. Cortinez, Raleigh's and Galerkin's methods: use of a variable parameter as multiplier versus minimization with respect to a exponential parameter. *J. Sound Vibr.* **124,** 388–389 (1988).
16. S. Wolfram Research, *Mathematica, Version* 2.2, Wolfram Research, Champaign, IL (1993).
17. M. A. De Rosa and C. Franciosi, The Rayleigh—Schmidt method and MATHEMATICA in vibrations and buckling problems. *J. Sound Vibr.* (in press).
18. J. Morrow, On the lateral vibration of bars supported at two points with one end overhanging. *Phil. Mag.* **15** (1908).
19. G. Krall, *Meccanica Tecnica delle Vibrazioni,* Eredi Virgilo Veschi, Roma (1970).

APPENDIX 1

The static deflection function of the beam in Fig. 1 is given by

$$v(x) = \frac{F}{6EI}\frac{N'}{D}$$

if $x \leqslant a$, and by

$$v(x) = \frac{F}{6EI}\frac{N''}{D}$$

otherwise, where

$$\begin{aligned}
N' = {} & 6a^3b^2T_1(3a^2 + 4ab + b^2 + 6a^2R_1 + 4abR_1 \\
& + 6a^2R_2 + 12abR_2 + 4b^2R_2 \\
& + 12a^2R_1R_2 + 12abR_1R_2 + 12abT_2 \\
& + 12b^2T_2 + 12abR_1T_2 + 12b^2R_2T_2 \\
& + 6a^2b^2R_1[a^2 + 2ab + b^2 + 2a^2R_2 + 6abR_2 \\
& + 4b^2R_2 - 6a^2T_1 - 12a^2R_2T_1 \\
& + 6abT_2 + 12b^2T_2 + 12b^2R_2T_2]x \\
& + 3ab^2[a^2 + 2ab + b^2 + 2a^2R_2 + 6abR_2 \\
& + 4b^2R_2 - 6a^2T_1 - 12a^2R_2T_1 \\
& + 6abT_2 + 12b^2T_2 + 12b^2R_2T_2]x^2 \\
& - b^2[3a^2 + 4ab + b^2 + 6a^2R_1 + 4abR_1 \\
& + 6a^2R_2 + 12abR_2 + 4b^2R_2 \\
& + 12a^2R_1R_2 + 12abR_1R_2 - 12abT_2 \\
& + 12b^2T_2 + 12abR_1T_2 + 12b^2R_2T_2]x^3
\end{aligned}$$

$$\begin{aligned}
N'' = {} & 2a^3b^3[a + b + R_1(4a + 3b) + R_2(3a + 4b) \\
& + 12R_1R_2(a + b) + 3T_1(4a + b) \\
& + 12R_1T_1a + 6R_2T_1(6a + 2b) + 36aR_1R_2T_1 \\
& + 3T_2(a + 4b) + 6R_1T_2(2a + 6b) \\
& + 12bR_2T_2 + 36bR_1R_2T_2 + 36T_1T_2(a + b) \\
& + 36aR_1T_1T_2 + 36bR_2T_1T_2] \\
& + 3a^2b^2[b^2 - a^2 - 2R_1(2a^2 - b^2) - 2R_2(a^2 - 2b^2) \\
& - 8R_1R_2(a^2 - b^2) - 12a^2T_1 - 12a^2R_1T_1 - 24a^2R_2T_1 \\
& - 24a^2R_1R_2T_1 + 12b^2T_2 + 24b^2R_1T_2 \\
& + 12b^2R_2T_2 + 24b^2R_1R_2T_2]x \\
& - 6a^2b^2[a + b + R_1(3a + 2b) + R_2(2a + 3b) \\
& + 6R_1R_2(a + b) + 3aT_1 \\
& + 6aR_2T_1 + 3bT_2 + 6bR_1T_2]x^2 \\
& + a^2[a^2 + 4ab + 3b^2 + 2R_1(2a^2 + 6ab + 3b^2) \\
& + 2R_2(2ab + 3b^2) + 12R_1R_2(ab + b^2) \\
& + 12T_1(a^2 + ab) + 12a^2R_1T_1 + 12abR_2T_1]x^3
\end{aligned}$$

$$D = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$+4R_1(a^4 + 3a^3b + 4a^2b^2 + ab^3)$$
$$+ 4R_2(a^3b + 3a^2b^2 + 4ab^3 + b^4)$$
$$+ 12R_1R_2(a^3b + 2a^2b^2 + ab^3)$$
$$+ 12T_1(a^4 + a^3b) + 12a^4R_1T_1$$
$$+ 12a^3bR_2T_1 + 12T_2(ab^3 + b^4)$$
$$+ 12ab^3R_1T_2 + 12b^4R_2T_2.$$

If this function is inserted into eqns (33) and (34), the Rayleigh quotient can be written as

$$\bar{\omega}^2 = EI\frac{N}{140a^2b^2D'm + D''M},$$

where

$$N = 420\{a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$+ R_1(4a^4 + 12a^3b + 12a^2b^2 + 4ab^3)$$
$$+ R_2(4a^3b + 12a^2b^2 + 12ab^3 + 4b^4)$$
$$+ R_1R_2(12a^3b + 24a^2b^2 + 12ab^3)$$
$$+ T_1(12a^4 + 12a^3b + 12a^4R_1 + 12a^3bR_2)$$
$$+ T_2(12ab^3 + 12b^4 + 12ab^3R_1 + 12b^4R_2)\}$$
$$\{a + b + R_1(4a + 3b) + R_2(3a + 4b)$$
$$+12R_1R_2(a + b) + 3T_1(4a + b + 4aR_1 + 12aR_2$$
$$+ 4bR_2 + 12aR_1R_2)$$
$$+ 3T_2[a + 4b + 4R_1(a + 3b) + 4bR_2$$
$$+ 12bR_1R_2 + 12T_1(a + b)$$
$$+ 12aR_1T_2 + 12bR_2T_1]\}$$

$$D' = (a + b)^2 + R_1(8a^2 + 14ab + 6b^2)$$
$$+ R_2(6a^2 + 14ab + 8b^2)$$
$$+ R_1^2(16a^2 + 24ab + 9b^2)$$
$$+ R_2^2(9a^2 + 24ab + 16b^2)$$
$$+ R_1^2R_2(96a^2 + 168ab + 72b^2)$$
$$+ R_1R_2^2(72a^2 + 168ab + 96b^2)$$
$$+ R_1R_2(48a^2 + 98ab + 48b^2)$$
$$+ 144R_1^2R_2^2(a + b)^2$$
$$+ T_1[(24a^2 + 30ab + 6b^2)$$
$$+ R_1(120a^2 + 120ab + 18b^2)$$
$$+ R_1^2(96a^2 + 72ab) + R_2(144a^2 + 210ab + 48b^2)$$
$$+ R_1R_2(720a^2 + 840ab + 144b^2)$$
$$+ R_1^2R_2(576a^2 + 504ab)$$
$$+ R_2^2(216a^2 + 360ab + 96b^2)$$
$$+ R_1R_2^2(1080a^2 + 1440ab + 288b^2)$$
$$+ R_1^2R_2^2(864a^2 + 864ab)]$$
$$+ T_1^2[(144a^2 + 72ab + 9b^2)$$
$$+ R_1(288a^2 + 72ab) + R_1^2 144a^2$$
$$+ R_2(864a^2 + 504ab + 72b^2)$$
$$+ R_1R_2(1728a^2 + 504ab)$$
$$+ R_1^2R_2 864a^2 + R_2^2(1296a^2 + 864ab + 144b^2)$$
$$+ R_1R_2^2(2592a^2 + 864ab) + 1296a^2R_1^2R_2^2]$$
$$+ T_2[(6a^2 + 30ab + 24b^2)$$
$$+ R_1(48a^2 + 210ab + 144b^2)$$
$$+ R_1^2(96a^2 + 360ab + 216b^2)$$
$$+ R_2(18a^2 + 120ab + 120b^2)$$
$$+ R_1R_2(144a^2 + 840ab + 720b^2)$$
$$+ R_1^2R_2(288a^2 + 1440ab + 1080b^2)$$
$$+ R_2^2(72ab + 96b^2) + R_1R_2^2(504ab + 576b^2)$$
$$+ R_1^2R_2^2(864ab + 864b^2)]$$
$$+ T_1T_2[(144a^2 + 450ab + 144b^2)$$
$$+ R_1(720a^2 + 1800ab + 432b^2)$$
$$+ R_1^2(576a^2 + 1080ab)$$
$$+R_2(432a^2 + 1800ab + 720b^2)$$
$$+ R_1R_2(2160a^2 + 7200ab + 2160b^2)$$
$$+ R_1^2R_2(1728a^2 + 4320ab) + R_2^2(1080ab + 576b^2)$$
$$+ R_1R_2^2(4320ab + 1728b^2) + R_1^2R_2^2 2592ab]$$
$$+ T_1^2T_2[(864a^2 + 1080ab + 216b^2)$$
$$+ R_1(1728a^2 + 1080ab) + R_1^2 864a^2$$
$$+ R_2(2592a^2 + 4320ab + 1080b^2)$$
$$+ R_1R_2(5184a^2 + 4320ab)$$
$$+ 2592a^2R_1^2R_2 + R_2^2(2592ab + 864b^2)$$
$$+ 2592abR_1R_2^2]$$
$$+ T_2^2[(9a^2 + 72ab + 144b^2)$$
$$+ R_1(72a^2 + 504ab + 864b^2)$$
$$+ R_1^2(144a^2 + 864ab + 1296b^2)$$
$$+ R_2(72ab + 288b^2) + R_1R_2(504ab + 1728b^2)$$
$$+ R_1^2R_2(864ab + 259b^2) + 144b^2R_2^2$$
$$+ R_1R_2^2(864b^2 + 1296b^2)]$$

$$+T_2^2T_1[(216a^2+1080ab+864b^2)$$
$$+R_1(1080a^2+4320ab+2592b^2)$$
$$+R_1^2(864a^2+2592ab)$$
$$+R_2(1080ab+1728b^2)+R_1R_2(4320ab+5184b^2)$$
$$+2592abR_1^2R_2+864b^2R_2^2+2592b^2R_1R_2^2]$$
$$+T_1^2T_2^2[(1296a^2+2592ab+1296b^2)$$
$$+R_1(2592a^2+2592ab)+1296a^2R_1^2$$
$$+R_2(2502ab+2592b^2)$$
$$+2592abR_1R_2+1296b^2R_2^2]$$

$$D''=3a^8b+22a^7b^2+68a^6b^3+115a^5b^4$$
$$+115a^4b^5+68a^3b^6+22a^2b^7+3ab^8$$
$$+R_1(33a^8b+220a^7b^2+612a^6b^3+920a^5b^4$$
$$+805a^4b^5+408a^3b^6+110a^2b^7+12ab^8)$$
$$+R_1^2(96a^8b+576a^7b^2+1424a^6b^3+1872a^5b^4$$
$$+1404a^4b^5+596a^3b^6+132a^2b^7+12ab^8)$$
$$+R_2(12a^8b+110a^7b^2+408a^6b^3+805a^5b^4$$
$$+920a^4b^5+612a^3b^6+220a^2b^7+33ab^8)$$
$$+R_1R_2(132a^8b+1100a^7b^2$$
$$+3672a^6b^3+6440a^5b^4$$
$$+6440a^4b^5+3672a^3b^6+1100a^2b^7+132ab^8)$$
$$+R_1^2R_2(384a^8b+2880a^7b^2$$
$$+8544a^6b^3+13104a^5b^4$$
$$+11232a^4b^5+5364a^3b^6+1320a^2b^7+132ab^8)$$
$$+R_2^2(12a^8b+132a^7b^2+596a^6b^3+1404a^5b^4$$
$$+1872a^4b^5+1424a^3b^6+576a^2b^7+96ab^8)$$
$$+R_1R_2^2(132a^8b+1320a^7b^2+5364a^6b^3$$
$$+11232a^5b^4+13104a^4b^5$$
$$+8544a^3b^6+2880a^2b^7+384ab^8)$$
$$+R_1^2R_2^2(384a^8b+3456a^7b^2$$
$$+12480a^6b^3+22848a^5b^4$$
$$+22848a^4b^5+12480a^3b^6+3456a^2b^7+384ab^8)$$
$$+T_1[(189a^8b+1008a^7b^2$$
$$+2184a^6b^3+2472a^5b^4$$
$$+1545a^4b^5+504a^3b^6+66a^2b^7)$$
$$+R_1(1512a^8b+7056a^7b^2$$
$$+13104a^6b^3+12360a^5b^4$$
$$+6180a^4b^5+1512a^3b^6+132a^2b^7)$$
$$+R_1^2(1764a^8b+7056a^7b^2+10920a^6b^3$$
$$+8208a^5b^4+2952a^4b^5+384a^3b^6)$$
$$+R_2(756a^8b+5040a^7b^2+13104a^6b^3+17304a^5b^4$$
$$+12360a^4b^5+4536a^3b^6+660a^2b^7)$$
$$+R_1R_2(6048a^8b+35280a^7b^2$$
$$+78624a^6b^3+86520a^5b^4$$
$$+49440a^4b^5+13608a^3b^6+1320a^2b^7)$$
$$+R_1^2R_2(7056a^8b+35280a^7b^2$$
$$+65520a^6b^3+57456a^5b^4$$
$$+23616a^4b^5+3456a^3b^3)$$
$$+R_2^2(756a^8b+6048a^7b^2$$
$$+19152a^6b^3+30240a^5b^4$$
$$+25200a^4b^5+10560a^3b^6+1728a^2b^7)$$
$$+R_1R_2^2(6048a^8b+42336a^7b^2$$
$$+114912a^6b^3+151200a^5b^4$$
$$+100800a^4b^5+31680a^3b^6+3456a^2b^7)$$
$$+R_1^2R_2^2(7056a^8b+42336a^7b^2+95670a^6b^3$$
$$+100800a^5b^4+48384a^4b^5+8064a^3b^6)]$$
$$+T_1^2[(6048a^8b+20160a^7b^2+25200a^6b^3$$
$$+14832a^5b^4+4212a^4b^5+468a^3b^6)$$
$$+R_1(18900a^8b+50400a^7b^2+47880a^6b^3$$
$$+19584a^5b^4+2952a^4b^5)+R_1^2(15120a^8b$$
$$+30240a^7b^2+20160a^6b^3+4752a^5b^4)$$
$$+R_2(24192a^8b+100800a^7b^2+151200a^6b^3$$
$$+103824a^5b^4+33696a^4b^5+4212a^3b^6)$$
$$+R_1R_2(75600a^8b+252000a^7b^2$$
$$+287280a^6b^3+137088a^5b^4+23616a^4b^5)$$
$$+R_1^2R_2(60480a^8b+151200a^7b^2$$
$$+120960a^6b^3+33264a^5b^4)+R_2^2(24192a^8b$$
$$+120906a^7b^2+221760a^6b^3+181440a^5b^4$$
$$+68544a^4b^5+9792a^3b^6)+R_1R_2^2(75600a^8b$$
$$+302400a^7b^2+423360a^6b^3+241920a^5b^4$$
$$+48384a^4b^5)+R_1^2R_2^2(60480a^8b+181440a^7b^2$$
$$+181440a^6b^3+60480a^5b^4)]$$
$$+T_2[(66a^7b^2+504a^6b^3+1545a^5b^4$$
$$+2472a^4b^5+2184a^3b^6+1008a^2b^7+189ab^8)$$

$$+ R_1(660a^7b^2 + 4536a^6b^3 + 12360a^5b^4$$
$$+ 17304a^4b^5 + 13104a^3b^6 + 5040a^2b^7 + 756ab^8)$$
$$+ R_1^2(1728a^7b^2 + 10560a^6b^3 + 25200a^5b^4$$
$$+ 30240a^4b^5 + 19152a^3b^6 + 6048a^2b^7 + 756ab^8)$$
$$+ R_2(132a^7b^2 + 1512a^6b^3 + 6180a^5b^4 + 12360a^4b^5$$
$$+ 13104a^3b^6 + 7056a^2b^7 + 1512ab^8)$$
$$+ R_1R_2(1320a^7b^2 + 13608a^6b^3 + 49440a^5b^4$$
$$+ 86520a^4b^5 + 78624a^3b^6 + 35280a^2b^7 + 6048ab^8)$$
$$+ R_1^2R_2(3456a^7b^2 + 31680a^6b^3$$
$$+ 100800a^5b^4 + 151200a^4b^5$$
$$+ 114912a^3b^6 + 42336a^2b^7 + 6048ab^8)$$
$$+ R_2^2(384a^6b^3 + 2952a^5b^4 + 8208a^4b^5$$
$$+ 10920a^3b^6 + 7056a^2b^7 + 1764ab^8)$$
$$+ R_1R_2^2(3456a^6b^3 + 23616a^5b^4 + 57456a^4b^5$$
$$+ 65520a^3b^6 + 35280a^2b^7 + 7056ab^8)$$
$$+ R_1^2R_2^2(8064a^6b^3 + 48384a^5b^4 + 100800a^4b^5$$
$$+ 95760a^3b^6 + 42336a^2b^7 + 7056ab^8)]$$
$$+ T_1T_2[(3024a^7b^2 + 16128a^6b^3 + 32760a^5b^4$$
$$+ 32760a^4b^5 + 16128a^3b^6 + 3024a^2b^7)$$
$$+ R_1(21168a^7b^2 + 96768a^6b^3 + 163800a^5b^4$$
$$+ 131040a^4b^5 + 48384a^3b^6 + 6048a^2b^7)$$
$$+ R_1^2(21168a^7b^2 + 80640a^6b^3 + 105840a^5b^4$$
$$+ 60480a^4b^5 + 12096a^3b^6)$$
$$+ R_2(6048a^7b^2 + 48384a^6b^3 + 131040a^5b^4$$
$$+ 63800a^4b^5 + 96768a^3b^6 + 21168a^2b^7)$$
$$+ R_1R_2(42336a^7b^2 + 290304a^6b^3 + 655200a^5b^4$$
$$+ 655200a^4b^5 + 290304a^3b^6 + 42336a^2b^7)$$
$$+ R_1^2R_2(42336a^7b^2 + 241920a^6b^3 + 423360a^5b^4$$
$$+ 302400a^4b^5 + 72576a^3b^6) + R_2^2(12096a^6b^3$$
$$+ 60480a^5b^4 + 105840a^4b^5 + 80640a^3b^6$$
$$+ 21168a^2b^7) + R_1R_2^2(72576a^6b^3$$
$$+ 302400a^5b^4 + 423360a^4b^5 + 241920a^3b^6$$
$$+ 42336a^2b^7) + R_1^2R_2^2(60480a^6b^3 + 181440a^5b^4$$
$$+ 181440a^4b^5 + 60480a^3b^6)]$$
$$+ T_1^2T_2[(60480a^7b^2 + 181440a^6b^3 + 196560a^5b^4$$
$$+ 90720a^4b^5 + 15120a^3b^6) + R_1(151200a^7b^2$$
$$+ 332640a^6b^3 + 241920a^5b^4 + 60480a^4b^5)$$
$$+ R_1^2(90720a^7b^2 + 120960a^6b^3 + 45360a^5b^4)$$
$$+ R_2(120960a^7b^2 + 544320a^6b^3 + 786240a^5b^4$$
$$+ 453600a^4b^5 + 90720a^3b^6) + R_1R_2(302400a^7b^2$$
$$+ 997920a^6b^3 + 97680a^5b^4 + 302400a^4b^5)$$
$$+ R_1^2R_2(181440a^7b^2 + 362880a^6b^3 + 181440a^5b^4)$$
$$+ R_2^2(120960a^6b^3 + 362880a^5b^4$$
$$+ 302400a^4b^5 + 75600a^3b^6) + R_1R_2^2(181440a^6b^3$$
$$+ 362880a^5b^4 + 181440a^4b^5)]$$
$$+ T_2^2[(468a^6b^3 + 4212a^5b^4 + 14832a^4b^5$$
$$+ 25200a^3b^6 + 20160a^2b^7 + 6048ab^8)$$
$$+ R_1(4212a^6b^3 + 33696a^5b^4 + 103824a^4b^5$$
$$+ 151200a^3b^6 + 100800a^2b^7 + 24192ab^8)$$
$$+ R_1^2(9792a^6b^3 + 68544a^5b^4 + 181440a^4b^5$$
$$+ 221760a^3b^6 + 120960a^2b^7 + 24192ab^8)$$
$$+ R_2(2952a^5b^4 + 19584a^4b^5 + 47880a^3b^6$$
$$+ 50400a^2b^7 + 18900ab^8) + R_1R_2(23616a^5b^4$$
$$+ 137088a^4b^5 + 287280a^3b^6 + 25200a^2b^7$$
$$+ 75600ab^8) + R_1^2R_2(48384a^5b^4 + 241920a^4b^5$$
$$+ 423360a^3b^6 + 302400a^2b^7 + 75600ab^8)$$
$$+ R_2^2(4752a^4b^5 + 20160a^3b^6 + 30240a^2b^7$$
$$+ 15120ab^8) + R_1R_2^2(33264a^4b^5 + 120960a^3b^6$$
$$+ 151200a^2b^7 + 60480ab^8) + R_1^2R_2^2(60480a^4b^5$$
$$+ 181440a^3b^6 + 181440a^2b^7 + 60480ab^8)]$$
$$+ T_1T_2^2[(15120a^6b^3 + 90720a^5b^4 + 196560a^4b^5$$
$$+ 181440a^3b^6 + 60480a^2b^7)$$
$$+ R_1(90720a^6b^3 + 453600a^5b^4 + 786240a^4b^5$$
$$+ 544320a^3b^6 + 120960a^2b^7) + R_1^2(75600a^6b^3$$
$$+ 302400a^5b^4 + 362880a^4b^5 + 120960a^3b^6)$$
$$+ R_2(60480a^5b^4 + 241920a^4b^5$$
$$+ 332640a^3b^6 + 151200a^2b^7)$$
$$+ R_1R_2(302400a^5b^4 + 967680a^4b^5$$
$$+ 997920a^3b^6 + 302400a^2b^7) + R_1^2R_1(181440a^5b^4$$
$$+ 362880a^4b^5 + 181440a^3b^6)$$
$$+ R_2^2(45360a^4b^5 + 120960a^3b^6 + 90720a^2b^7)$$
$$+ R_1R_2^2(181440a^4b^5 + 362880a^3b^6 + 181440a^2b^7)]$$

$$+ T_1^2 T_2^2 [(181440a^6b^3 + 544320a^5b^4 + 544320a^4b^5$$
$$+ 181440a^3b^6) + R_1(362880a^6b^3 + 725760a^5b^4$$
$$+ 362880a^4b^5) + R_1^2(181440a^6b^3 + 181440a^5b^4)$$
$$+ R_2(362880a^5b^4 + 725760a^4b^5 + 362880a^3b^6)$$
$$+ R_1 R_2(362880a^5b^4 + 362880a^4b^5)$$
$$+ R_2^2(181440a^4b^5 + 181440a^3b^6)].$$