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ON NATURAL BOUNDARY CONDITIONS AND DQM

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Introduction

The differential quadrature method (henceforth DQM) is a general purpose approach to approximate and numerically solve boundary and initial value problems. Quite recently, the method has been applied to a large number of structural systems, as beams, arches, plates and shells, and its scope has been greatly expanded. The main disadvantage of DQM seems to be the application of the boundary conditions in fourth–order equations, where more than one condition must be imposed at the same point. A considerable improvement with respect to the approximate δ–approach has been recently proposed by Chen et al. [1997].
In this paper, a simple device is proposed, which allows us to treat geometric and natural boundary conditions in an unified context. The procedure is applied to stability and dynamic analysis of beams, and the convergence rate of the results seems to be quite satisfactory.

The extended weighting coefficients

Let us consider a beam with span L, Young modulus E, second moment of area I, mass density ρ and cross sectional area A and let us define a cartesian reference coordinate x. It is convenient to adopt a natural coordinate system ξ, defined in the natural interval $[-1, 1]$, by means of the transformaion rule:

$$\xi(x) = 2\left(\frac{x}{L}\right) - 1 \tag{1}$$

The natural interval is divided into n segments defined by means of $n+1$ points located at the abscissae $\xi_1, \xi_2, \ldots, \xi_{n+1}$.
We shall assume the following set of $(n+7)$ nodal unknowns:

$$\mathbf{d}^T = \{u_1, \; u_1', \; u_1'', \; u_1''', \; u_2, \; \cdots, \; u_{n+1}, \; u_{n+1}', \; u_{n+1}'', \; u_{n+1}'''\} \tag{2}$$

i.e. the displacements at each nodal points, plus the first three derivatives at the end points.
Consequently, the displacement $v(\xi)$ of the beam can be approximed by:

$$v(\xi) = \boldsymbol{\alpha}\mathbf{C} = \sum_{i=1}^{n+7} \alpha_i C_i \tag{3}$$

where $\boldsymbol{\alpha}$ is a row vector of monomials, and $\mathbf{C}$ is a column vector of Lagrangian coordinates.
Two choices of the $\boldsymbol{\alpha}$ vector arise quite naturally:
1. the first elements of a power series:

$$\boldsymbol{\alpha} = (1 \quad \xi \quad \xi^2 \quad \cdots \quad \xi^{n+6}) \tag{4}$$

2. the first elements of the Chebyshev polynomials of the first kind:

$$\boldsymbol{\alpha} = (T_0(\xi) \quad T_1(\xi) \quad T_2(\xi) \quad \cdots \quad T_{n+6}(\xi)) \tag{5}$$

In the first case the nodal points will be simply located at equally spaced coordinates:

$$\xi_i = \frac{2(i-1)-n}{n}; \qquad i = 1, 2, \ldots, n+1 \tag{6}$$

whereas in the Chebyshev case it is convenient to use the so called Gauss-Lobatto-Chebyshev points:

$$\xi_i = -\cos\left(\frac{\pi(i-1)}{n}\right); \qquad i = 1, 2, \ldots, n+1 \tag{7}$$

From Eqn.(3) it is easily seen that:

$$\begin{aligned} v'(\xi) &= \alpha'\mathbf{C} \\ v''(\xi) &= \alpha''\mathbf{C} \\ v'''(\xi) &= \alpha'''\mathbf{C} \end{aligned} \tag{8}$$

and therefore Eqn.(3) and Eqn.(8) can be evaluated at the nodal points. It will be:

$$\mathbf{d} = \begin{Bmatrix} \alpha_1 \\ \alpha_1' \\ \alpha_1'' \\ \alpha_1''' \\ \alpha_2 \\ \vdots \\ \alpha_{n+1}''' \end{Bmatrix} = \mathbf{N}_0\mathbf{C} \tag{9}$$

Following the same approach as in Chen et. al. [1997] we define the weighting coefficients of the first four derivatives, as follows:

$$\mathbf{A} = \mathbf{N}_0'\mathbf{N}_0^{-1}; \quad \mathbf{B} = \mathbf{AA}; \quad \mathbf{C} = \mathbf{AAA}; \quad \mathbf{D} = \mathbf{AAAA} \tag{10}$$

<u>Frequency analysis</u>

The dynamic analysis of an elastic slender beam can be summarized by the following eigenvalue problem:

$$v''''(z) = \Omega^2 v(z) \tag{11}$$

where Ω is the nondimensional free vibration frequency of the system:

$$\Omega_i = \sqrt{\frac{\rho A \omega_i^2 L^4}{EI}} \tag{12}$$

and ω_i are the free vibration frequencies.
The discretized version of Eqn.(11) is given by:

$$\begin{pmatrix} D_{1,1} & D_{1,2} & \dots & D_{1,n+7} \\ D_{2,1} & D_{2,2} & \dots & D_{2,n+7} \\ D_{3,1} & D_{3,2} & \dots & D_{3,n+7} \\ D_{4,1} & D_{4,2} & \dots & D_{4,n+7} \\ \vdots & \vdots & \ddots & \vdots \\ D_{n+7,1} & D_{n+7,2} & \dots & D_{n+7,n+7} \end{pmatrix} \begin{pmatrix} v_1 \\ v_1' \\ v_1'' \\ v_1''' \\ \vdots \\ v_{n+1}''' \end{pmatrix} = \Omega^2 \begin{pmatrix} v_1 \\ v_1' \\ v_1'' \\ v_1''' \\ \vdots \\ v_{n+1}''' \end{pmatrix} \tag{13}$$

As an example, let us now consider a cantilever beam for which the boundary conditions are given by:

$$v(-1) = v'(-1) = v''(1) = v'''(1) = 0 \tag{14}$$

or, in discretized form:

$$v_1 = v_1' = v_{n+1}'' = v_{n+1}''' = 0 \tag{15}$$

It is now convenient to interchange the last two rows and the last two columns of the matrix $\mathbf{D}$ with the third and fourth rows and the third and fourth columns, so that the boundary conditions can be immediately imposed:

$$\left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ \hline D_{5,1} & D_{5,2} & D_{5,n+6} & D_{5,n+7} & D_{5,5} & \dots & D_{5,4} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ D_{4,1} & D_{4,2} & D_{4,n+6} & D_{4,n+7} & D_{4,5} & \dots & D_{4,4} \end{array}\right) \left(\begin{array}{c} v_1 \\ v_1' \\ v_{n+1}'' \\ v_{n+1}''' \\ \hline v_2 \\ \vdots \\ v_1''' \end{array}\right) = \Omega^2 \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline v_2 \\ \vdots \\ v_1''' \end{array}\right) \tag{16}$$

As can be seen, it is not necessary to perform any condensation, because it is sufficient to solve the reduced eigenvalue problem:

$$\begin{pmatrix} D_{5,5} & D_{5,6} & \dots & D_{5,3} & D_{5,4} \\ D_{6,5} & D_{6,6} & \dots & D_{6,3} & D_{6,4} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ D_{3,5} & D_{3,6} & \dots & D_{3,3} & D_{3,4} \\ D_{4,5} & D_{4,6} & \dots & D_{4,3} & D_{4,4} \end{pmatrix} \begin{pmatrix} v_2 \\ v_3 \\ \vdots \\ v_1'' \\ v_1''' \end{pmatrix} = \Omega^2 \begin{pmatrix} v_2 \\ v_3 \\ \vdots \\ v_1'' \\ v_1''' \end{pmatrix} \tag{17}$$

Actually, the row and column interchanges have been used only for the sake of clarity, but it is easy to realize that a more effective computational code can be based on row and column substitutions.
In order to check the numerical performance of the method, a simple *Mathematica* code was written, and in tables 1–2 the first six free vibration frequencies of the cantilever beam have been given, together with the exact results.

	Ω_1	Ω_2	Ω_3
n=4 Uniform grid	3.51601296	22.0404841	61.6227869
n=4 Chebyshev grid	3.51602018	22.0124366	67.9852392
n=5 Uniform grid	3.51601534	22.0338612	61.7987166
n=5 Chebyshev grid	3.51601529	22.0391356	61.4356717
n=6 Uniform grid	3.51601527	22.0345233	61.6924990
n=6 Chebyshev grid	3.51601527	22.0344419	61.8547408
n=7 Uniform grid	3.51601527	22.0345029	61.6972528
n=7 Chebyshev grid	3.51601527	22.0345039	61.6971933
n=10 Uniform grid	3.51601527	22.0344916	61.6971958
n=10 Chebyshev grid	3.51601525	22.0344916	61.6972192
n=14 Uniform grid	3.51601527	22.0344916	61.6972144
n=14 Chebyshev grid	3.51601471	22.0344918	61.6972144
Exact	3.51601527	22.0344916	61.6972144

TABLE 1: First three free vibration frequencies of a cantilever beam

As can be seen, the convergence rate is quite satisfactory, and the uniform grid behaves better than the Chebyshev grid, expecially for the higher frequencies. Moreover, we had no difficulties in obtaining the eigenvalues, whereas Bert and Malik [1996] reported some cases in which their convergence procedure failed. From this point of view, *Mathematica* allowed us to minimize the round–off errors, by introducing the numerical approximations only in the eigenvalue calculations. Finally, our convergence rate seems to be faster than the convergence rate given in the paper by Bert and Malik [1996, pag.12].

Stability analysis

Let us consider now the stability analysis in the presence of an axial compressive

$$v(-1) = v'(-1) = v(1) = v''(1) = 0 \tag{21}$$

or, in discretized form:

$$v_1 = v_1' = v_{n+1} = v_{n+1}'' = 0 \tag{22}$$

The third and fourth rows and columns must be interchanged with rows and columns (n+4) and (n+6), and subsequently the boundary conditions can be imposed immediately, leading to the reduced generalized eigenvalue problem:

$$\left[\begin{pmatrix} D_{5,5} & D_{5,6} & \cdots & D_{5,4} & D_{5,n+7} \\ D_{6,5} & D_{6,6} & \cdots & D_{6,4} & D_{6,n+7} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ D_{3,5} & D_{3,6} & \cdots & D_{3,4} & D_{3,n+7} \\ D_{n+5,5} & D_{n+5,6} & \cdots & D_{n+5,3} & D_{n+5,4} \\ D_{4,5} & D_{4,6} & \cdots & D_{4,4} & D_{4,n+7} \\ D_{n+7,5} & D_{n+7,6} & \cdots & D_{n+7,4} & D_{n+7,n+7} \end{pmatrix} - \lambda \begin{pmatrix} B_{5,5} & B_{5,6} & \cdots & B_{5,4} & B_{5,n+7} \\ B_{6,5} & B_{6,6} & \cdots & B_{6,4} & B_{6,n+7} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{3,5} & B_{3,6} & \cdots & B_{3,4} & B_{3,n+7} \\ B_{n+5,5} & B_{n+5,6} & \cdots & B_{n+5,3} & B_{n+5,4} \\ B_{4,5} & B_{4,6} & \cdots & B_{4,4} & B_{4,n+7} \\ B_{n+7,5} & B_{n+7,6} & \cdots & B_{n+7,4} & B_{n+7,n+7} \end{pmatrix}\right] \begin{pmatrix} v_2 \\ v_3 \\ \vdots \\ v_1'' \\ v_{n+1}' \\ v_1''' \\ v_{n+1}''' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{23}$$

In table 3 the nondimensional critical load is reported, together with the results given by Chen et al. [1997]. The present approach seems to be more precise, even if some care must be taken in order to perform the comparisons between the present method (n segments, $n + 7$ Lagrangian coordinates) and the Chen et al. procedure (n segments, $n + 4$ Lagrangian coordinates).

Conclusions

A modified differential quadrature method has been applied to stability and dynamic analysis of beams. The main advantage of the proposed approach seems to