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LETTERS TO THE EDITOR

NON-CLASSICAL BOUNDARY CONDITIONS AND DQM

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1. Introduction

The differential quadrature method (henceforth DQM) seems to be a promising numerical tool for analyzing differential equations with boundary and/or initial conditions.

At the early stage of the method, the satisfaction of the Neumann boundary conditions in fourth order systems was rather intriguing, and the so-called δ approach, as proposed by Bert and coworkers (see, e.g., reference [1]), turned out to be approximate and not completely reliable. Moreover, sometimes it produced badly conditioned matrices, with consequent numerical inaccuracies. More recently, an improved method [2] allowed one to satisfy exactly all the boundary conditions in a fourth order system, and a straightforward generalization of this approach [3] somewhat simplified the analysis in the presence of classical boundary conditions of the Dirichlet and Neumann type. Another, powerful generalization should also be mentioned [4, 5].

In this letter, the DQM is applied to dynamic and stability analysis of beams with non-classical boundary conditions, the obtained results are compared with the exact frequencies and critical loads, and the agreement is shown to be quite satisfactory for the entire parameter range.

2. The structural system

Consider a beam with span L, Young modulus E, second moment of area I, mass density ρ and cross-sectional area A. The beam ends are both elastically constrained against the vertical displacements and rotations, with vertical flexibilities at left and right c_{vl}, c_{vr}, respectively, and rotational flexibilities c_{rl}, c_{rr}.

The equation of motion of the beam in the presence of an axial force F at the right end can be written as

$$EI\, \partial^4 v/\partial x^4 + (F - k_p)\, \partial^2 v/\partial x^2 + k_w v - \rho A \omega^2 v = 0, \qquad (1)$$

where $v(x)$ is the transverse displacement, the parameters k_w and k_p define a two-parameter elastic soil, and ω^2 denotes the free vibration frequency of the beam.

The boundary conditions are

$$EI\, \partial^3 v/\partial x^3|_{x=0} = -(1/c_{vl}) v(0), \qquad EI\, \partial^2 v/\partial x^2|_{x=0} = (1/c_{rl})\, \partial v/\partial x|_{x=0},$$

$$EI\, \partial^3 v/\partial x^3|_{x=L} + (F - k_p)\, \partial v/\partial x|_{x=L} = (1/c_{vz}) v(L),$$

$$EI\, \partial^2 v/\partial x^2|_{x=L} = -(1/c_{rl})\, \partial v/\partial x|_{x=L}. \qquad (2)$$

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It is convenient to map the physical domain $[0, L]$ on to the natural Gaussian domain $[-1, 1]$, by means of the transformation

$$\xi(x) = 2(x/L) - 1, \tag{3}$$

where x is the Cartesian co-ordinate and ξ its natural counterpart.

It follows that the differential equation becomes

$$\partial^4 v(\xi)/\partial\xi^4 + (\lambda - \kappa_p)\,\partial^2 v(\xi)/\partial\xi^2 + \kappa_w v(\xi) - \Omega^2 v(\xi) = 0, \tag{4}$$

where

$$\lambda = FL^2/4EI, \qquad \kappa_p = k_p L^2/4EI, \qquad \kappa_w = k_w L^4/16EI, \qquad \Omega^2 = \rho A\omega^2 L^4/16EI. \tag{5}$$

The non-dimensional boundary conditions are given by

$$\partial^3 v/\partial\xi^3|_{\xi=-1} = -(1/\chi_{vl})v(-1), \qquad \partial^2 v/\partial\xi^2|_{\xi=-1} = (1/\chi_{rl})\,\partial v/\partial\xi|_{\xi=-1},$$

$$\partial^3 v/\partial\xi^3|_{\xi=1} + (\lambda - \kappa_p)\,\partial v/\partial\xi|_{\xi=1} = (1/\chi_{vz})v(1), \qquad \partial^2 v/\partial\xi^2|_{\xi=1} = -(1/\chi_{rl})\,\partial v/\partial\xi|_{\xi=1}, \tag{6}$$

where the non-dimensional axial flexibilities and rotational flexibilities can be expressed as

$$\chi_{vl} = 8EIc_{vl}/L^3, \qquad \chi_{rl} = 2EIc_{rl}/L, \qquad \chi_{vz} = 8EIc_{vz}/L^3, \qquad \chi_{rr} = 2EIc_{rr}/L. \tag{7}$$

3. A BRIEF OVERVIEW OF THE METHOD

In order to discretize the differential equation of motion, the natural interval is divided into n segments defined by means of $n+1$ points located at the abscissae $\xi_1, \xi_2, \ldots, \xi_{n+1}$. One can assume the set of $(n+7)$ nodal unknowns

$$\mathbf{d}^{\mathrm{T}} = \{u_1,\ u_1',\ u_1'',\ u_1''',\ u_2,\ \cdots,\ u_{n+1},\ u_{n+1}',\ u_{n+1}'',\ u_{n+1}'''\}, \tag{8}$$

and the displacement $v(\xi)$ of the beam can be approximated as

$$v(\xi) = \boldsymbol{\alpha}\mathbf{C} = \sum_{i=1}^{n+7} \alpha_i C_i, \tag{9}$$

where $\boldsymbol{\alpha}$ is a row vector of monomials, and $\mathbf{C}$ is a column vector of Lagrangian co-ordinates. From equation (9) it is easily seen that

$$v'(\xi) = \boldsymbol{\alpha}'\mathbf{C}, \qquad v''(\xi) = \boldsymbol{\alpha}''\mathbf{C}, \qquad v'''(\xi) = \boldsymbol{\alpha}'''\mathbf{C}, \tag{10}$$

and therefore

$$\mathbf{d} = \begin{Bmatrix} \alpha_1 \\ \alpha_1' \\ \alpha_1'' \\ \alpha_1''' \\ \alpha_2 \\ \vdots \\ \alpha_{n+1}''' \end{Bmatrix} = \mathbf{N}_0\mathbf{C}. \tag{11}$$

Following the same approach as in reference [1], one can define the weighting coefficients of the first four derivatives, as follows:

$$\mathbf{A} = \mathbf{N}_0'\mathbf{N}_0^{-1}, \qquad \mathbf{B} = \mathbf{AA}, \qquad \mathbf{C} = \mathbf{AAA}, \qquad \mathbf{D} = \mathbf{AAAA}. \tag{12}$$

The discretized version of equation (4) is

$$\begin{bmatrix} L_{1,1} & L_{1,2} & \cdots & L_{1,n+7} \\ L_{2,1} & L_{2,2} & \cdots & L_{2,n+7} \\ L_{3,1} & L_{3,2} & \cdots & L_{3,n+7} \\ L_{4,1} & L_{4,2} & \cdots & L_{4,n+7} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n+7,1} & L_{n+7,2} & \cdots & L_{n+7,n+7} \end{bmatrix} \begin{bmatrix} v_1 \\ v_1' \\ v_1'' \\ v_1''' \\ \vdots \\ v_{n+1}''' \end{bmatrix} = \Omega^2 \begin{bmatrix} v_1 \\ v_1' \\ v_1'' \\ v_1''' \\ \vdots \\ v_{n+1}''' \end{bmatrix} \tag{13}$$

where the matrix $\mathbf{L}$ is the discretized version of the differential operator

$$\mathscr{L} = \partial^4/\partial\xi^4 + (\lambda - \kappa_p)\,\partial^2/\partial\xi^2 + \kappa_w, \tag{14}$$

and, as such is given by

$$L_{ij} = D_{ij} + (\lambda - \kappa_p)B_{ij} + \kappa_w\delta_{ij}, \tag{15}$$

where δ_{ij} is the well-known Kronecker operator.

In order to impose the boundary conditions, it is now convenient to interchange the rows (and columns) $(n + 4)$ and $(n + 5)$ of the matrix $\mathbf{L}$ with the third and fourth rows (and columns), so that the boundary conditions can be immediately imposed:

$$\left[\begin{array}{cccc|cccccc} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & -\chi_{vl} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & \chi_{rl} & 0 & 0 & 0 \\ 0 & 0 & 1 & -\chi_{vz}(\lambda - \kappa_p) & 0 & \cdots & 0 & 0 & 0 & -\chi_{vz} \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & \chi_{rr} & 0 \\ \hline L_{5,1} & L_{5,2} & L_{5,n+4} & L_{5,n+5} & L_{5,5} & \cdots & L_{5,3} & L_{5,4} & L_{5,n+6} & L_{5,n+7} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ L_{3,1} & L_{3,2} & L_{3,n+4} & L_{3,n+5} & L_{3,5} & \cdots & L_{3,3} & L_{3,4} & L_{3,n+6} & L_{3,n+7} \\ L_{4,1} & L_{4,2} & L_{4,n+4} & L_{4,n+5} & L_{4,5} & \cdots & L_{4,3} & L_{4,4} & L_{4,n+6} & L_{4,n+7} \\ L_{n+6,1} & L_{n+6,2} & L_{n+6,n+4} & L_{n+6,n+5} & L_{n+6,5} & \cdots & L_{n+6,3} & L_{n+6,4} & L_{n+6,n+6} & L_{n+6,n+7} \\ L_{n+7,1} & L_{n+7,2} & L_{n+7,n+4} & L_{n+7,n+5} & L_{n+7,5} & \cdots & L_{n+7,3} & L_{n+7,4} & L_{n+7,n+6} & L_{n+7,n+7} \end{array}\right]$$

$$\times \left[\begin{array}{c} v_1 \\ v_1' \\ v_{n+1} \\ v_{n+1}' \\ \hline v_2 \\ \vdots \\ v_1'' \\ v_1''' \\ v_{n+1}'' \\ v_{n+1}''' \end{array}\right] = \Omega^2 \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline v_2 \\ \vdots \\ v_1'' \\ v_1''' \\ v_{n+1}'' \\ v_{n+1}''' \end{array}\right]. \tag{16}$$

In partitioned form, the previous equations can be written as

$$\begin{pmatrix} \mathbf{L}_{aa} & \mathbf{L}_{ab} \\ \mathbf{L}_{ba} & \mathbf{L}_{bb} \end{pmatrix} \begin{pmatrix} \mathbf{w}_c \\ \mathbf{w} \end{pmatrix} = \Omega^2 \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix}, \tag{17}$$

where $\mathbf{w}_c$ is the vector of the passive coordinates,

$$\mathbf{w}_c = \begin{Bmatrix} v_1 \\ v_1' \\ v_{n+1} \\ v_{n+1}' \end{Bmatrix}, \tag{18}$$

and $\mathbf{w}$ is the vector of the active co-ordinates,

$$\mathbf{w} = \begin{Bmatrix} v_2 \\ v_3 \\ \vdots \\ v_1'' \\ v_1''' \\ v_{n+1}'' \\ v_{n+1}''' \end{Bmatrix}. \tag{19}$$

The passive degrees of freedom can be easily condensed, and the following reduced eigenvalue problem is obtained:

$$(\mathbf{L}_{bb} - \mathbf{L}_{ba}\mathbf{L}_{aa}^{-1}\mathbf{L}_{ab})\mathbf{w} = \Omega^2\mathbf{w}. \tag{20}$$

It is perhaps worth noting that, in the absence of axial forces and elastic soil, no matrix inversion is involved in the condensation process, because in this case the matrix $\mathbf{L}_{aa}$ is given by an identity matrix.

4. NUMERICAL EXAMPLES

All computations for the numerical examples have been performed by using two different choices of the monomials α_i. In the first case $\alpha_i = \xi^{i-1}$ and the sampling points are uniformly distributed along the natural interval

$$\xi_i = [2(i-1) - n]/n, \qquad i = 1, 2, \ldots, n+1. \tag{21}$$

In the second case $\alpha_i = T_{i-1}(\xi)$, where $T_i(\xi)$ are the Chebyshev polynomials of the first kind, and the sampling points are located at the so-called Gauss–Lobatto–Chebyshev points,

$$\xi_i = -\cos(\pi(i-1)/n), \qquad i = 1, 2, \ldots, n+1. \tag{22}$$

In Table 1 the first three nondimensional natural frequencies of vibration are reported, in the absence of axial loads and elastic soils, for $\chi_{vz} = 0$, $\chi_{rl} = 0$, $\chi_{rr} = 0$, and for various values of the non-dimensional vertical flexibility at left χ_{vl}. The results have been obtained for $n = 8$, and are compared with the exact results, which in this particular case could be obtained by solving the frequency equation [6]. It is worth noting that the use of the Chebyshev polynomials implies a greater precision, especially for the higher frequencies. In any case, the agreement is quite satisfactory.

TABLE 1

First three non-dimensional frequencies of a beam with flexible ends

χ_{vl}		Ω_1	Ω_2	Ω_3
0	Uniform grid	22·37329	61·67194	120·85757
	Chebyshev grid	22·37329	61·67436	120·82882
	Exact	22·37329	61·67282	120·90339
0·001	Uniform grid	21·38780	53·11998	91·57713
	Chebyshev grid	21·38780	53·12383	91·52081
	Exact	21·38780	53·12356	91·51347
0·005	Uniform grid	17·75903	37·73705	77·64691
	Chebyshev grid	17·75903	37·73758	77·60824
	Exact	17·75901	37·73758	77·60118
0·01	Uniform grid	14·80957	33·89799	76·09249
	Chebyshev grid	14·80957	33·89823	76·05848
	Exact	14·80957	33·89823	76·05188
0·05	Uniform grid	8·884959	30·90089	74·94655
	Chebyshev grid	8·884959	30·90100	74·91615
	Exact	8·884959	30·90100	74·90992
0·1	Uniform grid	7·470370	30·55817	74·80982
	Chebyshev grid	7·470370	30·55827	74·77985
	Exact	7·470370	30·55827	74·77366
1	Uniform grid	5·813812	30·25851	74·68802
	Chebyshev grid	5·813812	30·25860	74·65844
	Exact	5·813812	30·25860	74·65230
10	Uniform grid	5·615816	30·22903	74·67590
	Chebyshev grid	5·615817	30·22912	74·64637
	Exact	5·615815	30·22912	74·64022
100	Uniform grid	5·595576	30·22608	74·67469
	Chebyshev grid	5·595575	30·22617	74·64516
	Exact	5·595575	30·22617	74·63902

In Table 2 the influence of the axial force on the free vibration frequency is illustrated for a beam clamped at the left and simply supported at the right with a flexible support ($\chi_{vl} = 0{\cdot}5$). Even in this case $n = 8$, whereas the support at the right is simulated by giving the large value $\chi_{rr} = 100\,000$ as the right rotational flexibility.

TABLE 2

Free vibration frequencies versus axial load for a propped cantilever beam with a flexible support

λ	Ω_1, uniform grid	Ω_1, exact
0	4·49483	4·49482
1	3·96342	3·96342
2	3·30980	3·30980
3	2·42482	2·42482
4	0·63363	0·63364
4·05	0·33936	0·33937
4·07	0·01667	0·01680
4·070047	0·00279	0·00347

The performances of the differential quadrature method are not influenced by the value of the axial loads, which can reach its critical value without causing any numerical error.

5. CONCLUSIONS

The differential quadrature method has been applied to a class of one-dimensional boundary problems in the presence of non-classical boundary conditions. It is shown that the proposed approach satisfies exactly all the four boundary conditions, leading to a simple eigenvalue problem.

A small *Mathematica* notebook [7] was written, in order to compare the appropriate eigenvalues with the exact results, and some effort was made in order to minimize the round-off errors. Finally, numerical examples and comparisons show the effectiveness of the proposed method.

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