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LIMIT BEHAVIOUR OF MASONRY ARCHES IN THE PRESENCE OF FINITE DISPLACEMENTS

CLAUDIO FRANCIOSI
Università di Napoli, Facoltà di Ingegneria, Istituto di Scienza delle Costruzioni, Piazzale Tecchio, Napoli 80125, Italy

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Abstract—A procedure is proposed for the calculation of the equilibrium paths of a masonry arch with elastic abutments. The elasticity of the abutments implies the existence of a non-linear, non-trivial equilibrium path, with the development of three hinges from the beginning of the load history. The shape of the equilibrium path suggests that shallow masonry arches should be classified in three different ways according to the nature of their failure load. In the first class a fourth hinge can develop as the live load is increased, and a classical failure load is reached when a four-hinge mechanism occurs. In the second case the equilibrium path shows a limit point, and finally it is possible to have a monotonically increasing equilibrium path, with no limit point or bifurcation point. If the arch is very shallow, only these two latter possibilities must be examined, and the presence of non-rigid abutments can cause instability of the arch. This instability phenomenon could be particularly important for flying buttresses of cathedrals, if they were built along rivers, or more generally on elastic soil.

NOTATION

a	span
a_1	length of the link AH
a_2	length of the link HK
b	rise
c_A, c_B	elastic flexibilities of the abutments
n	number of voussoirs
v_i	vertical displacement of the generic point i
w_i	horizontal displacement of the generic point i
w_A, w_B	horizontal displacements of the abutments
y_C, z_C	co-ordinates of the centre of rotation for the first link
A	index of the left hinge
B	index of the righthand section
H	index of the central hinge
H_A, H_B	horizontal total forces at the abutments
H_{Ag}, H_{Bg}	horizontal forces at the abutments due to the dead load
K	index of the righthand hinge
V_A, V_H, V_K	vertical reactions
β_1, β_2	angles between the links and the vertical line
δ	lagrangian co-ordinate, corresponding to total displacement of the abutments
δ^*	δ value corresponding to alignment of the hinges
ϕ_1, ϕ_2	rotations of the links AH and HK
ϕ_1^0, ϕ_2^0	non-trivial ϕ_1 and ϕ_2 values for $\delta = 0$

1. INTRODUCTION

In this paper the behaviour of a shallow masonry arch, subjected to the three Heyman hypotheses [1] is examined: sliding failure cannot occur; masonry has no tensile strength; and masonry has an infinite compressive strength. If the abutments are supposed to be rigid, then the arch behaviour resembles the behaviour of a rigid–plastic system: an increase of live loads causes no displacement up to a threshold value, where four hinges occur at the same moment. If no pattern of hinges can be obtained to lead to a four-bar chain, then the arch must be assumed 'perfect', and a failure load does not exist.

The aim of the present paper is to examine the behaviour of this kind of arch when the abutments are not perfectly rigid. This assumption changes qualitatively the arch behaviour, no matter how small the abutment displacement and the trivial equilibrium path is replaced by a non-linear equilibrium path.

The existence of non-zero displacements at the abutments implies the development of three hinges from the very beginning of the load history. It will be seen that there are two fundamentally different responses of this three-hinged arch. If the arch is not perfect, then as the live load is increased the hinges move and finally a fourth hinge can develop. In this way the Heyman multiplier can be obtained for the deflected structure, and usually a lower failure load than the classical one is obtained. If, on the contrary, the structure cannot fail according to Heyman, then the development of the fourth hinge is not observed, but usually the equilibrium path exhibits a limit point, with all associated snap-through phenomenon. It is obvious that in this case the limit load is the failure load. In other cases it can happen that the equilibrium path is an ever-increasing path that has no limit point or bifurcation point. The adjective 'perfect' will be used for this type of structure.

2. GEOMETRIC DESCRIPTION OF THE STRUCTURE

The arch is shown in Fig. 1, where the number of voussoirs is, say, n. In order to characterize the arch geometrically, the following information is needed:

(a) z and y values of the centre point of each assemblage section between two adjacent voussoirs;

(b) z and y values of the centre point of the middle section of each voussoir along the co-ordinate axes;

(c) the values of n vertical and n horizontal forces that are supposed to act at the centre of each voussoir;

(d) the height of each assemblage section;

(e) the two flexibilities c_A and c_B of the abutments.

There are no restrictive assumptions about the shape of the centre line of the arch, or about the loading system. Other important quantities, as for example span or rise, can be easily deduced from these data, as indicated in Fig. 1.

The simplest relationship between abutment displacement and reactive horizontal force is assumed

$$c_A H_A = w_A$$
$$c_B H_B = w_B, \tag{1}$$

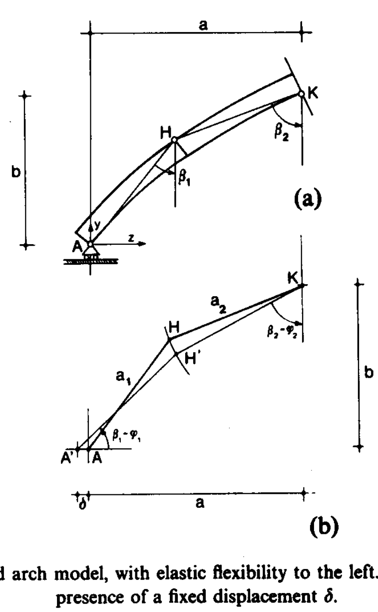

FIG. 1. (a) Three-hinged arch model, with elastic flexibility to the left. (b) Deflected structure in presence of a fixed displacement δ.

where w_A and w_B are the horizontal displacements of the abutments, and H_A and H_B are the reactive horizontal forces.

Suppose for the time being that the right abutment remains fixed, while the left abutment moves by the amount $\delta = w_A + w_B$. It is assumed, with no loss of generality, that the horizontal forces act from left to right, so that one of the hinges can be placed at the first section A. If it is assumed that the other two hinges develop at the generic sections H and K, then the arch can be considered as two rigid links AH and HK which connect the hinges.

The rotations ϕ_1 and ϕ_2 of these links must be expressed as a function of the assigned δ, if the arch has only one degree of freedom. A glance at Fig. 1(b) shows that it is possible to write

$$a_1 \cos(\beta_1 - \phi_1) + a_2 \cos(\beta_2 - \phi_2) = a_1 \cos\beta_1 + a_2 \cos\beta_2$$
$$a_1 \sin(\beta_1 - \phi_1) + a_2 \sin(\beta_2 - \phi_2) = a_1 \sin\beta_1 + a_2 \sin\beta_2. \quad (2)$$

This system can be solved numerically to obtain the unknowns ϕ_1 and ϕ_2. The resulting graphs $\phi_1(\delta)$ and $\phi_2(\delta)$ are sketched in Fig. 2. The computation is made easier by the previous knowledge of the values δ^*, ϕ_1^0 and ϕ_2^0. δ^* corresponds to aligned hinges, and can be easily obtained as (Fig. 3)

$$\delta^* = (a_1 + a_2)\cos\beta - \mathrm{a}, \quad (3)$$

where

$$\beta = \arcsin\left(\frac{b}{a_1 + a_2}\right). \quad (4)$$

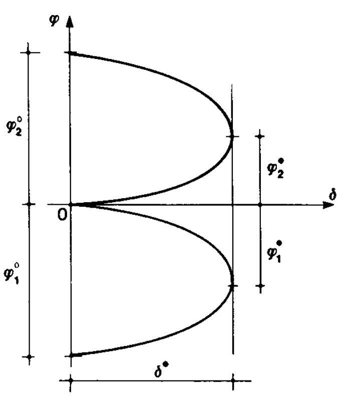

FIG. 2. Graphs $\phi_1(\delta)$ and $\phi_2(\delta)$.

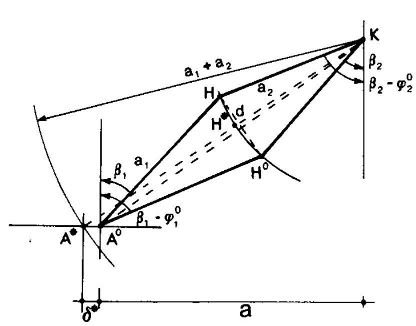

FIG. 3. Two extreme possibilities: the displacement δ corresponds to aligned hinges, the angles ϕ_1^0 and ϕ_2^0 correspond to 'inverted' structure at $\delta = 0$.

The non-trivial values ϕ_1^0 and ϕ_2^0 corresponding to $\delta = 0$ are shown in the same figure, and are calculated from the formulae:

$$\phi_1^0 = 4 \,\text{atn} \frac{(s-d)(s-a_1)}{s(s-a_2)}$$
$$\phi_2^0 = 4 \,\text{atn} \frac{(s-d)(s-a_2)}{s(s-a_1)}, \tag{5}$$

where

$$d = \sqrt{a^2+b^2} \tag{6}$$

and

$$s = \tfrac{1}{2}(d+a_1+a_2). \tag{7}$$

Hence the system has one degree of freedom, i.e. the total displacement δ of the abutments.

Once a δ value has been assigned, the rotations ϕ_1 and ϕ_2 of both the links AH and HK can be obtained, and then the vertical and horizontal displacements of every point of the arch can be calculated.

The hinge K remains fixed, and it is the centre of rotation for all the points of the link HK. Hence a generic point i, rigidly connected to this link, will have the following vertical and horizontal displacements [Fig. 4(b)]

$$v_i = d(\sin\alpha - \sin(\alpha+\phi_2))$$
$$w_i = d(\cos\alpha - \cos(\alpha+\phi_2)), \tag{8}$$

where

$$\alpha = \text{arctg}\frac{y_K - y_i}{z_K - z_i} \tag{9}$$

and

$$d = \sqrt{(y_K-y_i)^2+(z_K-z_i)^2}. \tag{10}$$

To calculate the displacements of the points of the link AH its rotation centre has to be found. The displacement of the hinge H is already known; hence the centre must lie on the straight line which connects K with the point $(y_H + v_H/2, z_H + w_H/2)$. On the other hand, the hinge A has by hypothesis no vertical displacement and a known horizontal displacement δ, so that the centre of rotation must lie on the vertical line of equation

$$z = -\frac{\delta}{2} + z_A. \tag{11}$$

The intersection of these two lines defines the centre of rotation C [Fig. 4(a)]

$$z_C = -\frac{\delta}{2} + z_A$$
$$y_C = y_K - \frac{(z_K - z_C)\left(y_K - \left(y_H + \frac{v_H}{2}\right)\right)}{z_K - \left(z_H + \frac{w_H}{2}\right)}. \tag{12}$$

Finally, [Fig. 4(c)],

$$v_i = d(\sin(\alpha+\phi_1) - \sin\alpha)$$
$$w_i = d(\cos(\alpha+\phi_1) - \cos\alpha), \tag{13}$$

where

$$\alpha = \text{arctg}\frac{y_C - y_i}{z_C - z_i} \tag{14}$$

and

$$d = \sqrt{(y_C-y_i)^2+(z_C-z_i)^2}. \tag{15}$$

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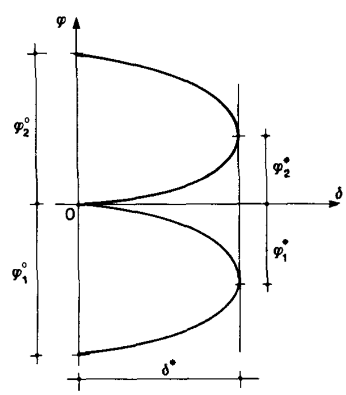

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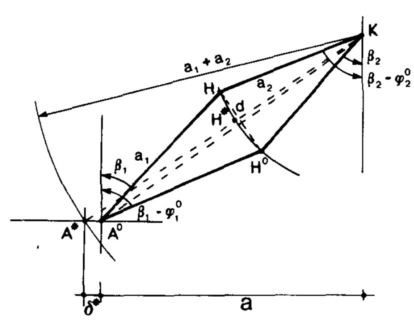

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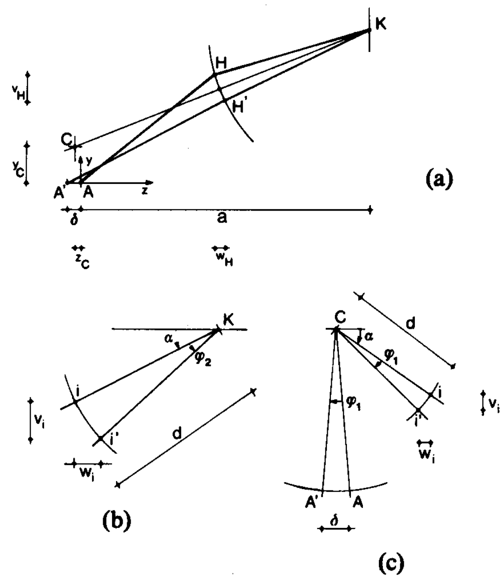

FIG. 4. (a) C is the centre of rotation for the points of the link AH, (b) displacements of the link HK, (c) displacements of the link AH.

3. EQUILIBRIUM RELATIONSHIPS AND THRUST LINE

After the calculation of the displacements of every assemblage section, the equilibrium equations of the deflected structure may be written. In particular the following quantities must be zero:

(a) sum of the vertical forces acting at the link AH;
(b) sum of the horizontal forces acting at the link AH;
(c) sum of the vertical forces acting at the link HK;
(d) sum of the horizontal forces acting at the link HK;
(e) sum of the moments at the hinge H of the forces acting at the link AH;
(f) sum of the moments at the hinge K of the forces acting at the link HK.

Another compatibility condition must be added to the above mentioned six equilibrium conditions

$$c_A(H_A - H_{Ag}) - c_B(H_B - H_{Bg}) = \delta. \tag{16}$$

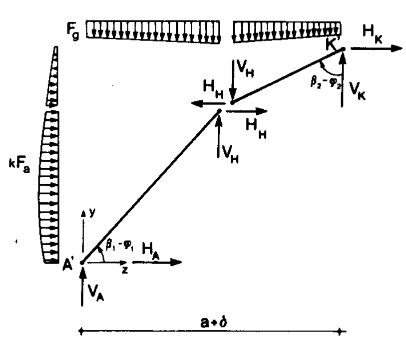

FIG. 5. Active and passive forces acting on the arch.

In this latter equation H_{Ag} and H_{Bg} are the horizontal forces at the abutments due to the dead load only. They are subtracted from the total forces H_A and H_B because it is supposed that the abutments move and the hinges develop only when the live load is applied. In other words, it is imagined that the arch was built in such a way that the voussoirs remain tight so long as the arch is subjected to the dead load only.

The six equilibrium conditions have to be written to accord with the nature of the live load. If for example the vertical load remains fixed, while the horizontal forces vary according to the multiplier k, then the following equations can be written (Fig. 5)

$$\begin{aligned}
&V_A+V_H=\sum_{i=1}^{H-1}F_{gi}\\
&H_A+H_H=\sum_{i=1}^{H-1}F_{ai}\\
&-V_H+V_K+\sum_{i=H}^{K}F_{gi}=0\\
&-H_H+H_K+k\sum_{i=H}^{K}F_{ai}=0\\
&H_A(y_H+v_H-y_A)+V_A(z_H+w_H-z_A-w_A)+k\sum_{i=1}^{H-1}F_{ai}(y_H+v_H-y_i-v_i)\\
&\qquad+\sum_{i=1}^{H-1}F_{gi}(z_H+w_H-z_i-w_i)=0\\
&-H_H(y_K-y_H-v_H)+V_H(z_K-z_H-w_H)+k\sum_{i=H}^{K}F_{ai}(y_K-y_i-v_i)\\
&\qquad+\sum_{i=H}^{K}F_{gi}(z_K-z_i-w_i)=0.
\end{aligned}\tag{17}$$

These equations can be solved, together with the compatibility equation, in order to obtain the equilibrium multiplier of the live load, and the unknown forces at the abutments.

It must be checked that the thrust line lies within the arch at each assemblage section. This may be accomplished by calculating the sum of all the forces acting at the left of the generic section and the sum of all the moments of these forces at the section. Then the ratio between the moment and the normal thrust gives the required position of the thrust line at the section.

In order to calculate accurately the normal thrust at the section, it is necessary to determine the angle of the deflected section with respect to the horizontal line. Thus, the horizontal displacements of the centre point and of the intrados point can be calculated at each section.

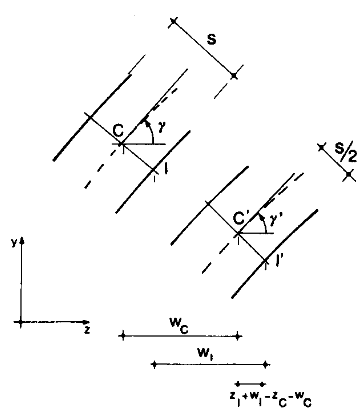

FIG. 6. Calculation of the angle γ' for the deflected structure. C is the centre of the section, I is the intrados point.

The angle is given by

$$\gamma' = \arccos\left((z_I + w_I - z_C - w_C)2/s\right), \tag{18}$$

where the meaning of the symbols is shown in Fig. 6.

4. THE EQUILIBRIUM PATH—SOME EXAMPLES

In the previous section δ and the position of the hinges were assigned, and the equilibrium multiplier k was found. If the thrust line lies in the arch at every assemblage section, then the point (δ, k) lies on the equilibrium path of the structure. If on the contrary there are sections in which the thrust line lies outside the arch, it is necessary to move the hinges. The best operative choice seems to put the hinges where the distance between the thrust line and the section is largest; in this way two or three steps are sufficient to obtain the actual positions of the hinges.

If the horizontal forces act from left to right it is easy to confirm that the left hinge A cannot move. Therefore only the position of the other two hinges must be checked.

The above procedure can be repeated for various δ values, until the equilibrium path is known with the desired accuracy. The proposed method has no incremental or iterative character, so that the path can be roughly sketched for some δ range, while it can be accurately drawn in other more interesting zones.

In the remaining part of this article some examples which show the various possible fates of shallow masonry arches will be illustrated.

(a) Consider the arch in Fig. 7. The span is 10 m, the rise is 2 m, the dead load is a constant distributed load equal to 3 t m^{-1}, while the live load is a (seismic) horizontal distributed load. The right abutment is supposed to be rigid, while the left abutment has the flexibility $c_A = 0.005$ m t^{-1}. Each voussoir is assumed 33 cm thick, and the arch is formed by 30 equal voussoirs. The classical Heyman calculation on the rigid structure leads to a failure multiplier equal to 2.4967. The hinges develop at both the abutments, at the fourth section, and at the eighteenth section.

The equilibrium path is shown in Fig. 8. It is a monotonically increasing path, up to $\delta = 18.5$ cm, where the procedure fails and convergence is never reached, because there is no compatible thrust line with only three hinges. In Fig. 9 the shapes of 10 thrust lines for 10 equidistant δ values are shown, from $\delta = 1.85$ to 18.5 cm. At the beginning the central hinge develops at the thirteenth section, while the righthand hinge is at the abutment. As δ is increased, both the hinges move from right to left, and finally the central hinge arrives at the fourth section, while the righthand hinge arrives at the eighteenth section. At this δ value the thrust line reaches the extrados at the right abutment, and no further increase of δ is possible. The value of the equilibrium multiplier at $\delta = 18.5$ cm is slightly lower than the classical failure load, and is equal to 2.33.

(b) An idealized flying buttress is shown in Fig. 10(a). The span is equal to 10 m, the rise is 1 m and the righthand abutment is 9 m higher than the left one. The arch is rather thick, i.e. 1 m, hence it must be considered 'perfect' according to Heyman. The lefthand abutment has a small flexibility $c_B = 0.0001$ m t^{-1}, while the righthand abutment has the flexibility $c_A = 0.02$ m t^{-1}. The dead load is a constant distributed load equal to 3 t m^{-1}, while the live

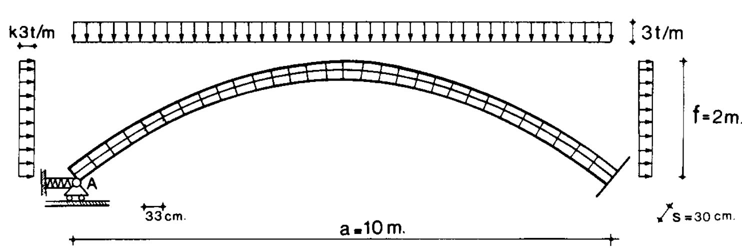

FIG. 7. First example. The classical failure load of the undeflected structure is 2.49, the failure load of the deflected structure is 2.33.

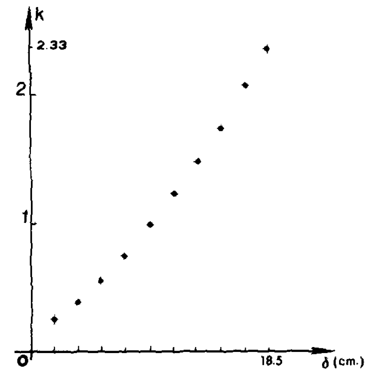

FIG. 8. Equilibrium path of the arch in Fig. 7.

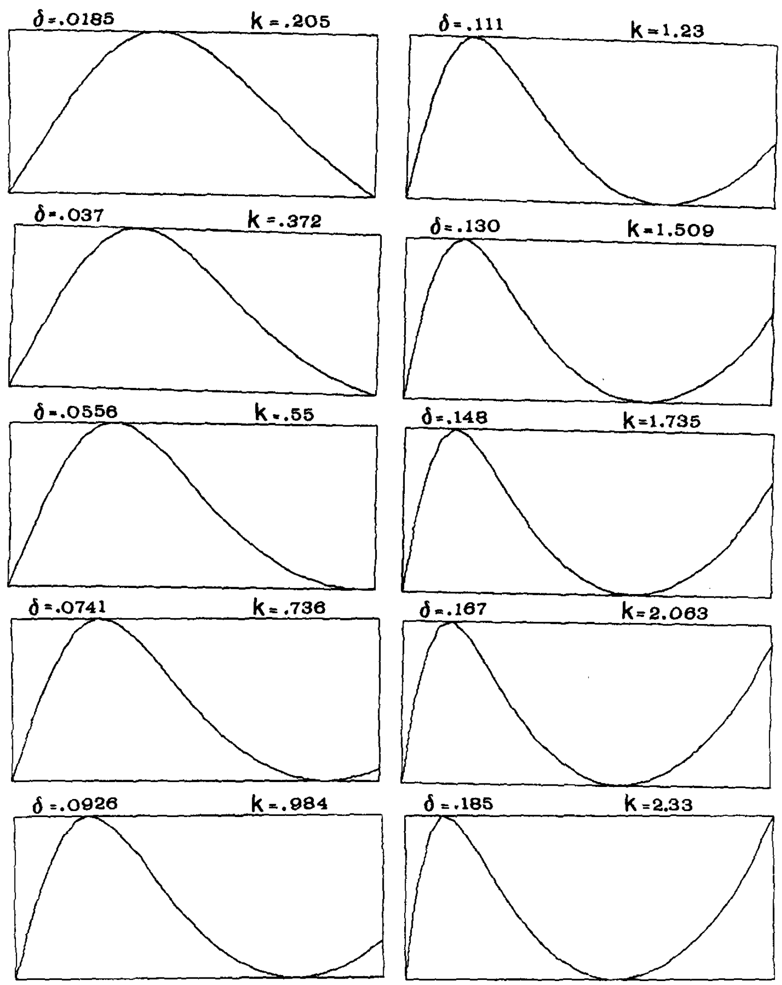

FIG. 9. Ten thrust lines of the arch in Fig. 7 corresponding to 10 different δ values.

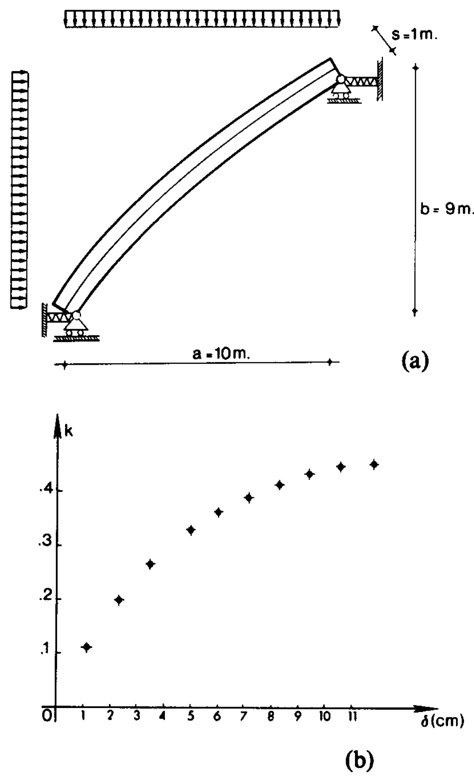

FIG. 10. Second example. (a) This arch should be considered 'perfect' according to Heyman, (b) the equilibrium path shows a limit point at $\delta \simeq 11.5$ cm.

load is a horizontal distributed load subjected to the multiplier k. The equilibrium path is sketched in Fig. 10(b), and the fate of the arch is immediately seen. All the load multipliers are negative, hence the live loads act from right to left. When k reaches the value -0.45 the lefthand abutment has a displacement equal to 11 cm and a snap-through phenomenon is incipient. No further load increase is possible, and the value $k = -0.45$ has to be assumed as failure load for this flying buttress.

CONCLUSIONS

A finite-displacement analysis of masonry arches allows the thorough examination of the behaviour of this type of structure in some interesting ranges of the ratio rise : span. The abutments are supposed to displace according to a simple linear relationship between reactive forces and relative displacements. If the classical failure load exists, it can be calculated for the deflected structure, and usually a lower coefficient than the classical one is obtained. If the classical failure load does not exist, a limit point can be reached, where the arch has a snap-through instability. Finally, for small elastic flexibilities, the Heyman hypothesis is verified, and no critical load exists.

Acknowledgement—The author would like to express his gratitude to Professor J. Heyman for clarifying the presentation of this work.

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