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HIGHER ORDER TIMOSHENKO QUOTIENT IN THE STABILITY AND DYNAMIC ANALYSIS OF SMOOTHLY TAPERED BEAMS

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It is well-known that the Timoshenko quotient always gives better results than the corresponding Rayleigh quotient, but its implementation is not straightforward, at least for non-uniform redundant beams. Quite recently a modified approach has been proposed [1], in which the main difficulty is overcome, and some preliminary results were given for a tapered beam. In this paper an iterative procedure is suggested, which leads to closer approximations to the true results, and to dramatic improvements in the Rayleigh quotient performances. Consequently, narrow lower-upper bounds can be deduced. Clamped beams and clamped-supported beams with rectangular cross-sections and linearly varying height are thoroughly investigated, providing some interesting comparisons with the results given in [1]. The exact differential equations have been solved for this particular cross-section variation law, in terms of Bessel functions, so that exact critical loads and free frequencies can be used to illustrate the performance of the proposed approach. A small program was written, by using the symbolic package Mathematica, so that a large sample of numerical examples could be offered.

1. INTRODUCTION

The semianalytical approach to buckling and vibration problems seems to share the advantages of analytical and numerical methods, without their drawbacks. In fact, the SAN method allows one to solve virtually all the structural problems which can be numerically solved, and the results will undoubtedly be more general than in the purely numerical case. On the other hand, the analytical solution is attainable only for particular systems, and its usefulness seems to be limited to comparisons with more general methods.

The most famous SAN result goes back to Lord Rayleigh, who gave an approximate formula for the upper bound to the first free vibration frequency of an elastic conservative system. Since then, its idea was generalized to cover stability problems, and more general eigenvalue problems for conservative systems.

A major step was undertaken by the same Rayleigh, who proposed a powerful optimization procedure. Unfortunately, the method turns out to be non-linear in nature, so that its practical implementation was unfeasible until the advent of powerful computers; only recently this improvement was re-discovered by Schmidt [2], and it is now usually referred to as the Rayleigh-Schmidt method.

A very useful companion to the Rayleigh quotient was proposed by Timoshenko [3], who basically used the total complementary energy method, whereas Rayleigh always started from the total potential energy of the system. Timoshenko used its quotient to

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deduce the critical loads for a large number of beam problems, and heuristically proved that its quotient should give better results than the corresponding Rayleigh formula. The main drawback of the Timoshenko quotient lies in the difficulty in deducing the bending moment from the trial deflection shape, and apparently Timoshenko confined himself to statically determinate beams, for which the problem can be trivially solved.

Later on, the Timoshenko quotient was extended by Ku [4] to statically indeterminate beams, and the inequality

$$t \leqslant r, \tag{1}$$

where t is the Timoshenko quotient and r is the corresponding Rayleigh quotient, was rigorously stated as a consequence of the well-known Cauchy-Schwarz inequality (see reference [4]).

More recently, Bhat [5] used the Timoshenko quotient in order to deduce the free vibration frequencies, and finally Cortinez and Laura [1] suggested an important improvement and makes it easier to use the Timoshenko quotient, regardless of the boundary conditions.

In this paper a general procedure is suggested, which is aimed at refining the eigenvalue prediction by using a sophisticated trial function.

The interesting point lies in the possibility of generating this function automatically, starting from the first choice, and taking advantage of the particular boundary conditions, cross-section variation law and loading distribution.

2. THE ITERATIVE METHOD

Consider an Euler-Bernoulli beam, with span l, Young modulus E, mass density ρ, cross sectional area $A(z)$, and second moment of area $I(z)$.

If the beam is subjected to an axial force P, then let $w(z)$ be the deflection shape, and $m(z)$ the corresponding bending moment, so that the Timoshenko quotient can be written as $P = A/B$, where

$$A = \int_0^l \frac{M^2(z)}{EI(z)}\,dz \quad B = \int_0^l w'^2(z)\,dz. \tag{2}$$

If the beam is subjected to inertia forces

$$q(z) = -\rho A(z)\omega^2 w(z), \tag{3}$$

then the Timoshenko quotient can be expressed as $\omega^2 = A/C$, where:

$$C = \int_0^l \rho A w^2(z)\,dz. \tag{4}$$

Obviously, the usefulness of the quotient lies in its capability in approximating critical loads and free frequencies, whenever the true deformed shape $w(z)$ is approximated by a trial function $\bar{w}(z)$ which satisfies at least the geometrical boundary conditions.

It is known that the Timoshenko quotient is less sensitive to the choice of the approximating function than the Rayleigh quotient. In other words, a poor choice of the trial function will result in a satisfactory upper bound, if the Timoshenko approach is adopted, whereas the Rayleigh quotient could lead to an appreciably overestimated eigenvalue.

If a refined result is required, then the following procedure is suggested.

1. Starting from a trial function, calculate the bending moment $m(z)$ by means of the Ku method [2], or by using the Cortinez-Laura approach.
2. Calculate the first approximation of the Timoshenko quotient.
3. Integrate twice the bending moment $m(z)$, *disregarding* the integration constant. In this way a function $g(z)$ is obtained.
4. Add to this function an *ad hoc* polynomial, in order to satisfy the boundary conditions.
5. Use this function as the new trial deflection shape.

It will be shown that two iterations lead to interesting improvements of the results, and that the Rayleigh quotient is often dramatically near to the Timoshenko quotient, if the iterated deflection shape is used. In turn, this fact leads to narrow lower-upper bounds, since it is known that the Hanna-Michalopoulos lower bound [6] is strongly influenced by the discrepancy between the Rayleigh quotient and the corresponding Timoshenko quotient.

In order to illustrate in detail this method, a calculation is presented of the critical load of a clamped beam with unit span, and whose inertia is assumed to vary according to the law

$$I(z) = I_0(1 + 0{\cdot}9z)^3. \tag{5}$$

It will be shown in the sequel that the exact non-dimensional critical load is equal to 105·8716.

1. The simplest trial function is given by

$$w_1(z) = z^2(z - 1)^2, \tag{6}$$

The Rayleigh quotient can be immediately used, and a first approximation to the critical load is obtained as

$$r_1 \approx 147{\cdot}4215EI_0/l^2, \tag{7}$$

The Ku procedure [2] allows one to deduce the bending moment due to an unit axial force as

$$m_1(z) = w_1(z) - 0{\cdot}03569z - 0{\cdot}01736, \tag{8}$$

and then (2) a trivial calculation shows that the Timoshenko quotient is equal to

$$t_1 \approx 119{\cdot}664EI_0/l^2. \tag{9}$$

(The method recently proposed by Cortinez and Laura [1] can also be used, and actually should be preferred, because it leads to the same quotient as above, and it is simpler to implement.) Here, however, t_1 is about 13% higher than the true result, so that a better approximation would seem to be useful.

3. The bending moment is then integrated twice, and the result is divided by $I(z)$. In this way the function $g(z)$ is obtained, after disregarding the integration constants:

$$g(z) = (0{\cdot}03333z^6 - 0{\cdot}1z^5 + 0{\cdot}08333z^4 - 0{\cdot}005948z^3 - 0{\cdot}00868z^2)/(1 + 0{\cdot}9z)^3 \tag{10}$$

4. This function does not satisfy the boundary conditions at $z = 1$. By adding to it the polynomial

$$az^2 + bz^3, \tag{11}$$

the constants a and b can be chosen to fulfill the required boundary conditions. It is not difficult to show that the second trial function is finally obtained as:

$$w_2(z) = g(z) + 0{\cdot}00158699z^2 - 0{\cdot}0012897z^3 \tag{12}$$

5. If this function is used, then the Timoshenko quotient becomes:

$$t_2 \approx 106{\cdot}4414EI_0/l^2 \tag{13}$$

which is just 0·5% higher than the exact value.

The corresponding Rayleigh quotient is easily calculated as:

$$r_2 \approx 108{\cdot}0190EI_0/l^2, \tag{14}$$

and it turns to be 2% higher than the exact critical load.

A lower bound can be deduced by using the Hanna-Michalopoulos formula [6]:

$$l_2 = t_2 - \sqrt{t_2(r_2 - t_2)/3} \approx 98{\cdot}96EI_0/l^2. \tag{15}$$

A third step leads to a Timoshenko quotient $t_3 \approx 105{\cdot}8803EI_0l^2$, to a close Rayleigh quotient $r_3 \approx 106{\cdot}0511EI_0l^2$, and to the corresponding lower bound $l_3 \approx 103{\cdot}425EI_0l^2$. A useful discussion about lower-upper bounds can be found in reference [7].

3. AN EXACT SOLUTION

Consider a beam with rectangular cross-section and linearly varying height, so that area and inertia will vary according to the laws

$$A(z) = A_0(1 + \beta z/l), \qquad I(z) = I_0(1 + \beta z/l)^3. \tag{16}$$

The equation of motion can be written as

$$(\partial^2/\partial z^2)[EI(z)\,\partial^2 w(z,t)/\partial z^2] + \rho A(z)\,\partial^2 w(z,t)/\partial t^2 = 0 \tag{17}$$

and its solution can be taken as

$$w(z,t) = V(z)\,\mathrm{e}^{i\omega t}, \tag{18}$$

so that the equation of motion becomes

$$(\mathrm{d}^2/\mathrm{d}z^2)[EI(z)\,\mathrm{d}^2V(z,t)/\mathrm{d}z^2] - \rho A(z)\omega^2V(z) = 0. \tag{19}$$

By taking into account equations (16), it is possible to write

$$u^2V''''(u) + 6uV'''(u) + 6V''(u) - (q/2)^4V(u) = 0, \tag{20}$$

with

$$u = 1 + \beta z/l, \qquad q = 2\lambda l/\beta, \qquad \lambda = (\rho A_0\omega^2/EI_0)^{1/4}, \tag{21}$$

and (′) denoting differentiation with respect to u.

The general solution of equation (20) can be expressed as

$$V(u) = (1/\sqrt{u})\{A\mathrm{J}_1[q\sqrt{u}] + B\mathrm{Y}_1[q\sqrt{u}] + C\mathrm{I}_1[q\sqrt{u}] + D\mathrm{K}_1[q\sqrt{u}]\}, \tag{22}$$

where J_1, Y_1, I_1 and K_1 are Bessel functions and modified Bessel functions of first order, and A, B, C, D are integration constants.

Two boundary conditions will be considered, corresponding to clamped-clamped and clamped-supported beams. In the first case the frequency equation will result by zeroing the determinant

$$\begin{vmatrix} J_1[q] & Y_1[q] & I_1[q] & K_1[q] \\ -J_2[q] & -Y_2[q] & I_2[q] & -K_2[q] \\ J_1[q\sqrt{u}] & Y_1[q\sqrt{u}] & I_1[q\sqrt{u}] & K_1[q\sqrt{u}] \\ -J_2[q\sqrt{u}] & -Y_2[q\sqrt{u}] & I_2[q\sqrt{u}] & -K_2[q\sqrt{u}] \end{vmatrix} = 0, \qquad (23)$$

whereas in the second case the boundary conditions lead to

$$\begin{vmatrix} J_1[q] & Y_1[q] & I_1[q] & K_1[q] \\ -J_2[q] & -Y_2[q] & I_2[q] & -K_2[q] \\ J_1[q\sqrt{u}] & Y_1[q\sqrt{u}] & I_1[q\sqrt{u}] & K_1[q\sqrt{u}] \\ J_3[q\sqrt{u}] & Y_3[q\sqrt{u}] & I_3[q\sqrt{u}] & K_3[q\sqrt{u}] \end{vmatrix} = 0. \qquad (24)$$

For the stability analysis the differential equation is

$$[EI(z)w'']'' + Pw'' = 0, \qquad (25)$$

or else

$$EI(z)w'' = m, \qquad EI(z)m'' + Pm = 0. \qquad (26)$$

Upon taking into account the variation law of the cross-sectional inertia, the second of equations (26) becomes

$$m'' + (Pl^2/EI_0)(1/\beta^2u^3)m = 0, \qquad (27)$$

or else

$$m'' + (p/u^3)m = 0. \qquad (28)$$

This is a Bessel equation, which can be solved as

$$m(u) = -A\sqrt{u}J_1[2\sqrt{p}/\sqrt{u}] - B\sqrt{u}Y_1[2\sqrt{p}/\sqrt{u}] \qquad (29)$$

Finally, one has:

$$w(u) = m(u) + Cu + D. \qquad (30)$$

The four integration constants have to be defined by imposing the boundary conditions. For the clamped-clamped beam the critical load parameter p will be obtained by satisfying the equation

$$\begin{vmatrix} -J_1[2\sqrt{p}] & -Y_1[2\sqrt{p}] & 1 & 1 \\ -\sqrt{p}J_2[2\sqrt{p}] & -\sqrt{p}Y_2[2\sqrt{p}] & 1 & 0 \\ -\sqrt{u}J_1[2\sqrt{p}/\sqrt{u}] & -\sqrt{u}Y_1[2\sqrt{p}/\sqrt{u}] & u & 1 \\ -\sqrt{p/u}J_2[2\sqrt{p}/\sqrt{u}] & -\sqrt{p/u}Y_2[2\sqrt{p}/\sqrt{u}] & 1 & 0 \end{vmatrix} = 0, \qquad (31)$$

whereas in the clamped-supported case it is possible to arrive at:

$$\begin{vmatrix} -J_1[2\sqrt{p}] & -Y_1[2\sqrt{p}] & 1 & 1 \\ -\sqrt{p}J_2[2\sqrt{p}] & -\sqrt{p}Y_2[2\sqrt{p}] & 1 & 0 \\ -\sqrt{u}J_1[2\sqrt{p}/\sqrt{u}] & -\sqrt{u}Y_1[2\sqrt{p}/\sqrt{u}] & u & 1 \\ 2(\sqrt{p}/u^2)J_2[2\sqrt{p}/\sqrt{u}] - (p/u^2\sqrt{u})J_3[2\sqrt{p}/\sqrt{u}] & 2(\sqrt{p}/u^2)Y_2[2\sqrt{p}/\sqrt{u}] - (p/u^2\sqrt{u})Y_3[2\sqrt{p}/\sqrt{u}] & 0 & 0 \end{vmatrix} = 0. \qquad (32)$$

In all the cases a bisection routine can calculate the eigenvalue within machine precision.

4. NUMERICAL RESULTS

The proposed approach has a general range of validity, and it can be potentially applied to beams with every kind of boundary conditions, and with generally varying cross-sections.

In order to illustrate its capabilities, clamped-clamped beams and clamped-supported beams with rectangular cross-section and linearly varying heights will be examined (cf. equation 16). In this way numerical comparisons with exact results can be performed, by solving the frequency equations and the critical load equations deduced in the previous section.

In Table 1 the critical loads of a clamped-clamped beam are given, with β allowed to vary between $-0{\cdot}9$ and $0{\cdot}9$. In the second column the exact results are given, as obtained by zeroing the determinant, in the third column the Timoshenko quotient is given, as obtained by using the simplest trial function (cf. equation 6), and in the fifth column the corresponding Rayleigh quotient is shown. (It is worth noting that the Timoshenko quotient can be obtained by using the Ku method [4] or the Cortinez-Laura suggestion [1], but this latter approach should be preferred because of its intrinsic simplicity. In order to stress this fact, in the Appendix the general formula for obtaining the critical loads is reported, for every β value.) In the fourth and sixth columns the iterated Timoshenko quotient and Rayleigh quotient are given.

As can be easily seen, the use of more refined functions leads to noticeable improvements of the Timoshenko quotient precision, at least for high β values, and these improvements are even more pronounced for the Rayleigh quotient. Consequently, as already said, the Hanna-Michalopoulos lower bound increases, and a satisfactory lower-upper bound can be deduced.

In Table 2 the critical loads for clamped-simply supported beams are given, and in Tables 3 and 4 the free vibration frequencies are reported for clamped-clamped and clamped-supported beams, respectively. The same qualitative behaviour as in Table 1 is observed.

TABLE 1

Critical loads for clamped-clamped beams, for different taper ratios

β	Exact result	Timoshenko 1st approx.	Timoshenko 2nd approx.	Rayleigh 1st approx.	Rayleigh 2nd approx.
−0·9	1·6700	6·0064	2·3313	14·3385	4·4283
−0·8	4·0853	7·9927	4·7434	15·4080	6·5326
−0·7	7·0449	10·406	7·6210	16·7895	9·1343
−0·6	10·479	13·235	10·864	18·564	11·915
−0·5	14·349	16·513	14·455	20·812	14·789
−0·4	18·626	20·229	18·654	23·616	18·747
−0·3	23·291	24·414	23·319	27·056	23·397
−0·2	28·330	29·088	28·354	31·212	28·424
−0·1	33·729	34·276	33·754	36·167	33·833
0	39·478	40·000	39·508	42·000	39·600
0·1	45·570	46·284	45·604	48·794	45·710
0·2	51·995	53·153	52·037	56·628	52·160
0·3	58·749	60·630	58·802	65·585	58·954
0·4	65·825	68·740	65·899	75·744	66·103
0·5	73·217	77·509	73·327	87·188	73·625
0·6	80·922	86·961	81·091	99·996	81·544
0·7	88·935	97·120	89·193	114·25	89·890
0·8	97·253	108·01	97·641	130·03	98·701
0·9	105·87	119·66	106·44	147·42	108·019

TABLE 2

Critical loads for clamped-supported beams, for different taper ratios

−0.0	Exact result	Timoshenko 1st approx.	Timoshenko 2nd approx.	Rayleigh 1st approx.	Rayleigh 2nd approx.
−0·9	0·8748	2·3114	1·3806	8·0063	3·4848
−0·8	2·1189	3·6113	2·6127	8·6400	4·3333
−0·7	3·6344	5·0519	4·0804	9·4238	5·5045
−0·6	5·3884	6·6611	5·7550	10·380	6·8777
−0·5	7·3622	8·4512	7·5311	11·531	8·0443
−0·4	9·5434	10·431	9·5605	12·900	9·6280
−0·3	11·923	12·607	11·975	14·509	12·166
−0·2	14·494	14·986	14·499	16·380	14·522
−0·1	17·252	17·574	17·255	18·536	17·263
0	20·142	20·377	20·207	21·000	20·243
0·1	23·308	23·401	23·343	23·794	23·423
0·2	26·600	26·651	26·661	26·940	26·802
0·3	30·063	30·133	30·163	30·461	30·391
0·4	33·697	33·853	33·853	34·380	34·206
0·5	37·498	37·815	37·736	38·719	38·268
0·6	41·465	42·027	41·819	43·500	42·605
0·7	45·596	46·492	46·112	48·746	47·248
0·8	49·889	51·218	50·626	54·480	52·238
0·9	54·343	56·208	55·377	60·724	57·629

TABLE 3

Free frequencies for clamped-clamped beams, for different taper ratios

β	Exact result	Timoshenko 1st approx.	Timoshenko 2nd approx.	Rayleigh 1st approx.	Rayleigh 2nd approx.
−0·9	9·8846	11·107	10·061	17·687	11·336
−0·8	11·842	12·660	11·919	17·554	12·552
−0·7	13·483	14·038	13·518	17·606	13·848
−0·6	14·962	15·331	14·970	17·839	15·101
−0·5	16·336	16·572	16·346	18·248	16·463
−0·4	17·634	17·777	17·676	18·821	17·968
−0·3	18·879	18·954	18·908	19·544	19·111
−0·2	20·078	20·111	20·090	20·400	20·171
−0·1	21·241	21·251	21·244	21·374	21·262
0	22·373	22·376	22·373	22·445	22·373
0·1	23·480	23·489	23·480	23·614	23·498
0·2	24·563	24·591	24·573	24·855	24·638
0·3	25·628	25·684	25·649	26·160	25·797
0·4	26·674	26·769	26·713	27·522	26·985
0·5	27·705	27·847	27·769	28·931	28·210
0·6	28·722	28·917	28·818	30·381	29·483
0·7	29·726	29·982	29·864	31·868	30·814
0·8	30·718	31·041	30·910	33·385	32·213
0·9	31·700	32·095	31·956	34·929	33·693

TABLE 4

Free frequencies for clamped-supported beams, for different taper ratios

β	Exact result	Timoshenko 1st approx.	Timoshenko 2nd approx.	Rayleigh 1st approx.	Rayleigh 2nd approx.
−0·9	8·6300	9·3027	8·8611	13·618	10·473
−0·8	9·7995	10·235	9·9045	13·396	10·754
−0·7	10·737	11·026	10·788	13·319	11·247
−0·6	11·556	11·747	11·579	13·366	11·813
−0·5	12·300	12·422	12·303	13·521	12·353
−0·4	12·990	13·064	13·014	13·768	13·220
−0·3	13·640	13·681	13·667	14·095	13·915
−0·2	14·258	14·277	14·265	14·491	14·336
−0·1	14·849	14·856	14·850	14·945	14·862
0	15·418	15·419	15·418	15·451	14·418
0·1	15·969	15·970	15·970	16·000	15·983
0·2	16·503	16·509	16·508	16·587	15·556
0·3	17·023	17·037	17·033	17·206	17·144
0·4	17·530	17·556	17·549	17·853	17·756
0·5	18·026	18·066	18·057	18·523	18·400
0·6	18·511	18·569	18·559	19·214	19·088
0·7	18·987	19·064	19·057	19·924	19·829
0·8	19·455	19·552	19·551	20·649	20·635
0·9	19·914	20·034	20·045	21·517	21·387

5. CONCLUSIONS

Starting from a poor trial function, a method for generating more refined trial functions has been devised, which allows some improvements in Timoshenko quotients and even more in Rayleigh quotients.

This method has been applied both to stability analysis and dynamic analysis of tapered beams, and numerical results have been presented for clamped beams and propped cantilever beams.

In order to perform numerical comparisons, exact stability and dynamic analyses of tapered rectangular beams with linearly varying height have been given, by solving the difference equations in terms of Bessel functions.

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APPENDIX

In order to deduce the first-order Timoshenko quotient for tapered rectangular beams with linearly varying height, a small symbolic program was written, which essentially reproduces the simplified theory given by Cortinez and Laura [1].

The non-dimensional quotient can always be expressed as:

$$t_1 = (A/B)(EI_0/l^2), \tag{33}$$

where, for the critical loads of clamped-clamped beams,

$$A = 12(2\beta^{11} - 2\beta^{10} \log(1+\beta) - \beta^{11} \log(1+\beta)), \tag{34}$$

$$\begin{aligned} B = 7(&6480\beta^3 + 16\,200\beta^4 + 13\,920\beta^5 + 4680\beta^6 + 512\beta^7 + 16\beta^8 - 6480\beta^2 \log(1+\beta) \\ &- 19\,440\beta^3 \log(1+\beta) - 21\,480\beta^4 \log(1+\beta) - 10\,560\beta^5 \log(1+\beta) \\ &- 2232\beta^6 \log(1+\beta) - 192\beta^7 \log(1+\beta) - 3\beta^8 \log(1+\beta) \\ &- 6480\beta \log(1+\beta)^2 - 22\,680\beta^2 \log(1+\beta)^2 - 30\,240\beta^3 \log(1+\beta)^2 \\ &- 18\,900\beta^4 \log(1+\beta)^2 - 5400\beta^5 \log(1+\beta)^2 - 540\beta^6 \log(1+\beta)^2 \\ &+ 6480 \log(1+\beta)^3 + 25\,920\beta \log(1+\beta)^3 + 41\,040\beta^2 \log(1+\beta)^3 \\ &+ 32\,400\beta^3 \log(1+\beta)^3 + 13\,140\beta^4 \log(1+\beta)^3 \\ &+ 2520\beta^5 \log(1+\beta)^3 + 180\beta^6 \log(1+\beta)^3). \end{aligned} \tag{35}$$

for the critical loads of clamped-supported beams,

$$A = 432(2\beta^{10} - \beta^{11} - 2\beta^9 \log(1+\beta)), \tag{36}$$

$$\begin{aligned} B = (7(&103\,680\beta^2 + 349\,920\beta^3 + 435\,720\beta^4 + 243\,120\beta^5 + 55\,546\beta^6 + 1302\beta^7 - 75\beta^8 \\ &- 207\,360\beta \log(1+\beta) - 803\,520\beta^2 \log(1+\beta) - 11\,90640\beta^3 \log(1+\beta) \\ &- 831\,960\beta^4 \log(1+\beta) - 268\,416\beta^5 \log(1+\beta) - 30\,750\beta^6 \log(1+\beta) \\ &+ 103\,680 \log(1+\beta)^2 + 453\,600\beta \log(1+\beta)^2 + 780\,840\beta^2 \log(1+\beta)^2 \\ &+ 660\,960\beta^3 \log(1+\beta)^2 + 278\,640\beta^4 \log(1+\beta)^2 \\ &+ 51\,840\beta^5 \log(1+\beta)^2 + 3240\beta^6 \log(1+\beta)^2), \end{aligned} \tag{37}$$

and finally, for the free frequencies of clamped-clamped beams,

$$A = -5\,544\,000(4\beta^{11} + 2\beta^{12} - 4\beta^{10} \log(1+\beta) - 4\beta^{11} \log(1+\beta) - 2\beta^{12} \log(1+\beta)), \tag{38}$$

$$\begin{aligned} B = (&-1\,293\,600\beta - 5\,821\,200\beta^2 - 8\,710\,240\beta^3 - 3\,320\,240\beta^4 + 1\,447\,600\beta^5 \\ &- 1\,663\,200\beta^6 - 2\,244\,088\beta^7 + 517\,748\beta^8 + 14\,574\beta^9 - 249\,431\beta^{10} \\ &+ 1\,293\,600 \log(1+\beta) + 6\,468\,000\beta \log(1+\beta) + 11\,513\,040\beta^2 \log(1+\beta) \\ &+ 7\,244\,160\beta^3 \log(1+\beta) - 304\,920\beta^4 \log(1+\beta) + 896\,280\beta^5 \log(1+\beta) \\ &+ 3\,195\,500\beta^6 \log(1+\beta) + 412\,720\beta^7 \log(1+\beta) - 360\,920\beta^8 \log(1+\beta) \\ &+ 315\,140\beta^9 \log(1+\beta) + 90\,390\beta^{10} \log(1+\beta)) \end{aligned} \tag{39}$$

and finally, for the free frequencies of clamped-supported beams,

$$A = 1\,386\,000\beta^{10}(76 + 43\beta)(-2\beta + \beta^2 + 2\log(1+\beta)), \tag{40}$$

$$\begin{aligned} B = (&-10\,348\,800\beta - 64\,680\,000\beta^2 - 139\,665\,680\beta^3 - 100\,221\,660\beta^4 + 14\,144\,130\beta^5 \\ &- 22\,135\,960\beta^6 - 90\,856\,381\beta^7 - 20\,828\,500\beta^8 + 11\,805\,220\beta^9 - 10\,984\,680\beta^{10} \\ &- 8\,578\,425\beta^{11} + 10\,348\,800\log(1+\beta) + 69\,854\,400\beta\log(1+\beta) \\ &+ 171\,143\,280\beta^2\log(1+\beta) + 165\,095\,700\beta^2\log(1+\beta) \\ &+ 26\,749\,800\beta^4\log(1+\beta) + 11\,018\,700\beta^5\log(1+\beta) + 104\,658\,400\beta^6\log(1+\beta) \\ &+ 62\,991\,390\beta^7\log(1+\beta) - 7\,142\,100\beta^8\log(1+\beta) + 5\,916\,750\beta^9\log(1+\beta) \\ &+ 14\,414\,400\beta^{10}\log(1+\beta) + 3\,346\,200\beta^{11}\log(1+\beta)). \end{aligned} \tag{41}$$