Free vibrations of foundation beams on Green soil in the presence of conservative and nonconservative axial loads

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In this paper a general dynamic analysis of a foundation beam on Green-Boussinesq soil is performed, taking into account the instabilizing effect of conservative and nonconservative applied axial loads.

The beam is reduced to a finite number of rigid bars, linked together by elastic springs; the equations of motion are written by means of the Lagrange equations. The kinetic energy and the total potential energy are calculated first, and emphasis is placed to the strain energy of the Green soil; then the virtual work of the applied follower loads is detected, which allow us to define the generalized forces. The resulting equations of motion lead to an eigenvalue problem with unsymmetric matrix.

Initially, the first free vibration frequencies of simply supported beams, clamped beams and free beams are plotted as functions of the two soil parameters. A more complex beam is also examined, in order to show the method potentialities. A stability analysis in the presence of conservative axial loads is then performed, and the influence of the soil on the critical load is discussed, both for simply supported beams and clamped beams. Finally, the instability mechanism of a clamped – clamped beam subjeced to a uniformly distributed follower force is shown to be deeply influenced by the presence of the soil. (A number of graphs and examples conclude the paper.)

1. INTRODUCTION

The aim of the present note is to study the free vibrations of a foundation beam resting on a Green soil, taking into account the influence of the applied axial forces, both of conservative and nonconservative nature.

The beam is supposed to be constrained in quite a general way, by introducing m elastically flexible supports and n elastically flexible hinges (Fig. 1), so that it is possible to immediately recover every particular system.

The soil behaviour is assumed to follow the Boussinesq-Green model; according to it a distributed load $r(z')\,dz'$ centered at the abscissa z' causes a displacement graph $v(z)$ which can be expressed as:

$$v(z) = c(z, z')\,dz'. \tag{1}$$

The function $c(z, z')$ is usually called the 'influence line' (or the Green function) of the displacements due to the forces, and completely defines the adopted soil model.

The applied forces can be divided into conservative and follower forces. To the first set of forces a potential energy can be associated, which is a quadratic function of the Lagrangian coordinates, and leads to a symmetric matrix. The follower forces, on the other hand, cannot be associated to any potential energy, by definition. Their influence on the free vibration frequencies can be taken into account by calculating their virtual work due to a virtual variation of the Lagrangian coordinates. Following this path, they give rise to an unsymmetric matrix.

The structure is studied according to the cells discretization method, in which the beam is reduced to t rigid bars, connected by $n = t + 1$ elastic springs ('cells'). The elastic strain energy of the rigid bars is supposed to be concentrated at the cells, by defining the following local stiffness coefficients (Fig. 2):

$$k_1' = \frac{EI_1}{2a}$$

$$k_i' = \frac{1}{2}\,\frac{EI_i + EI_{i-1}}{a} \qquad i = 2, \ldots n-1 \tag{2}$$

$$k_n' = \frac{EI_n}{2a}$$

where a is the length of the rigid bars, and I_i is the second moment of area of the cross section at the middle point of the ith bar. For the sake of simplicity, it was assumed that each rigid bar has equal length a, but this is by no means mandatory.

The extensional flexibility c_i of the elastic cells must also be introduced, in order to simulate the real constraints.

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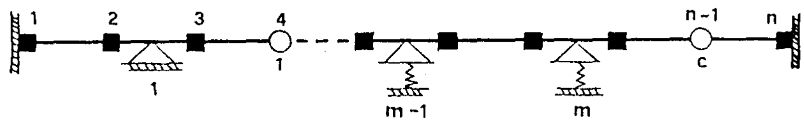

Fig. 1. The structural system, with n elastic cells, c hinges and m elastic support

The total flexibility f_i of the ith cell is given by:

$$f_i = \frac{1}{k_i^l + c_i} \tag{3}$$

and the total stiffness coefficient k_i is given by $1/f_i$. In this way it is possible to place a hinge at the abcissa $z = ai$ by defining $c_i = \infty$, whereas a clamped end corresponds to $c_1 = 0$ or $c_n = 0$, respectively. The elastically flexible supports are defined by their axial stiffness s_i, in such a way that $s_i = \infty$ corresponds to perfect support, whereas finite stiffness values give a flexible support. If all the axial stiffness s_i are equal to zero, then the axial forces acting on the beam must be self-equilibrated.

The mass of the beam is lumped at a finite number of sections. If, for example, the masses are concentrated at the cells abscissae, then the ith mass is given by:

$$m_i = \frac{\mu_{i-1} + \mu_i}{2} a$$
$$m_1 = \frac{\mu_1}{2} a \tag{4}$$
$$m_n = \frac{\mu_n}{2} a$$

where $\mu(z)$ is the mass density function, and μ_i is the mass density at the middle point of the ith rigid bar.

Finally, the Green function $c(z, z')$ can also be discretized, by defining its values at the centre of the t rigid bars. In this way, a square matrix $\mathbf{C}$ is generated, which can be defined as 'Green displacement matrix' of the soil. Its C_{ij} term represents the displacement of the ith cell due to an unitary force at the jth cell. The matrix $\mathbf{C}$ is obviously symmetric and positive definite. Its inverse – let us call it $\mathbf{R}$ – is the 'Green force matrix' of the soil. Its generic term r_{ij} is the force at the ith cell due to an unitary displacement at the jth cell.

The applied load distribution will be accordingly reduced to a set of n concentrated forces at the cells abscissae. Moreover, each force will be divided into its conservative part P^i and its nonconservative (follower) part F_i, so that the global external loading is reduced to a couple of n dimensional vectors $\mathbf{P}$ and $\mathbf{F}$.

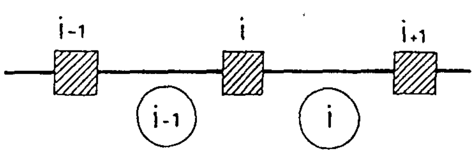

Fig. 2. Local stiffness coefficient

Once the previous discretizations are carried out, the structure is reduced to a finite degree of freedom holonomic system. The n Lagrangian coordinates can be arbitrarily chosen; for exmaple it is possible[1] to assume as Lagrangian coordinates the n displacements of the n cells. However, in the following we shall assume as Lagrangian coordinates the rotations of the rigid bars, and the displacement of the first cell, as already done in Ref. 2.

2. THE EQUATIONS OF MOTION

In this paragraph the kinetic energy will be calculated, together with the strain energy and the potential energy of the conservative applied loads. The virtual work of the follower loads will allow us to define the corresponding generalized forces, so that a straightforward application of the Lagrange equations will permit to deduce the equations of motion. The solution of these differential equations lead to an eigenvalue problem, which gives the free vibration frequencies and the free vibration modes.

2.1 The kinetic energy

If $\mathbf{v}$ is the n-dimensional vector of the transverse displacements, then the kinetic energy is a quadratic function of the velocities:

$$T = \frac{1}{2} \sum_i^n m_i \dot{v}_i^2 \tag{5}$$

which can be written conveniently as:

$$T = ½\dot{\mathbf{v}}^T\mathbf{M}\dot{\mathbf{v}} \tag{6}$$

where $\mathbf{M}$ is the (diagonal) mass matrix.

If $\mathbf{c}$ is the n-dimensional vector of the Lagrangian coordinates:

$$\mathbf{c}^T = (\phi_1, \ldots \phi_{n-1}, \eta) \tag{7}$$

then the relationship between $\mathbf{v}$ and $\mathbf{c}$ can be expressed as:

$$\mathbf{v} = Vc \tag{8}$$

where $\mathbf{V}$ is a square (n, n) matrix, given by

$$V_{ij} = a \quad i = 1, \ldots n; \quad j = 1, \ldots i - 1$$
$$V_{n,j} = 1 \quad j = 1, \ldots n \tag{9}$$

It follows that the kinetic energy has to be written as:

$$T = ½\dot{\mathbf{c}}^T\mathbf{V}^T\mathbf{M}\mathbf{C}\dot{\mathbf{c}} = ½\dot{\mathbf{c}}^T\tilde{\mathbf{M}}\dot{\mathbf{c}} \tag{10}$$

where $\tilde{\mathbf{M}}$ is a full square matrix.

2.2 The strain energy

The strain energy L is a quadratic function of the Lagrangian coordinates:

$$L = ½c^TKc \tag{11}$$

where **K** is the global stiffness matrix. The strain energy is due to the bending strain energy L_f of the cells, to the extensional strain energy L_e of the supports, and to the strain energy L_s of the soil. Every mutual energy is zero, so that:

$$L = L_f + L_e + L_s \tag{12}$$

The bending energy L_f can be easily expressed as:

$$L_f = \frac{1}{2}\sum_{i=1}^{n} k_i\psi_i^2 \tag{13}$$

where ψ_i is the relative rotation of the ith cell. It is:

$$\psi_1 = \phi_1$$
$$\psi_i = \phi_i - \phi_{i-1} \quad i = 2, \ldots n-1 \tag{14}$$
$$\phi_n = 1\phi_{n-1}$$

and consequently:

$$L_f = ½c^TK_fc. \tag{15}$$

$\mathbf{k_f}$ is a n-dimensional three-diagonal matrix, whose elements are given by:

$$K_{ii} = k_i + k_{i+1} \qquad i = 1, \ldots n-2 \tag{16}$$
$$K_{i,i+1},\ K_{i,i+1} = -k_{i+1} \qquad i = 1, \ldots n-2 \tag{17}$$
$$K_{n-1,n-1} = k_n \tag{18}$$
$$k_{i,n},\ K_{n,i} = 0 \qquad i = 1, \ldots n \tag{19}$$

The extensional strain energy of the m flexible supports is given by:

$$L_e = \frac{1}{2}\sum_{i=1}^{m} s_i d_i^2 \tag{20}$$

where s_i is the axial stiffness of the ith support, and d_i is the corresponding displacement.

If this support is placed on the jth rigid bar, it will be (Fig. 3):

$$d_j = \sum_{k=1}^{j-1} \phi_k a + \phi_j a' \tag{21}$$

Finally, the strain energy of the Green soil can be expressed as[1]:

$$L_s = \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} r_{ij}\int_0^a v_i(s)v_j(s)\,ds \tag{22}$$

On the other hand, it is:

$$v_i(s) = v_i + \frac{v_{i+1} - v_i}{a}s \tag{23}$$

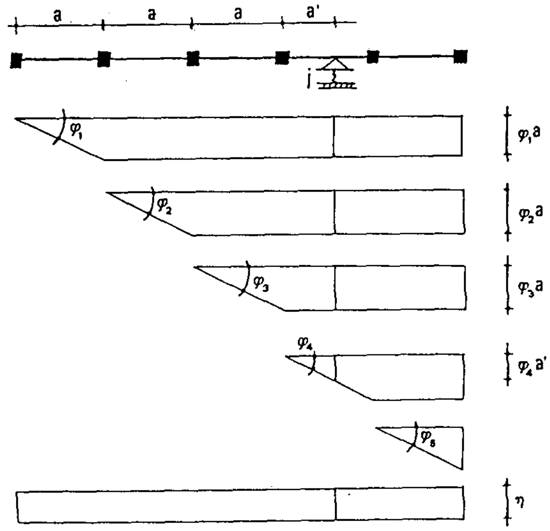

Fig. 3. The Lagrangian coordinates

After some algebra, it is possible to arrive to:

$$L_p = \frac{a}{2}\sum_{i=1}^{n-1} r_{ii}\frac{v_i^2 + v_{i+1}^2 + v_iv_{i+1}}{3} + \frac{a}{2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} r_{ij}$$
$$\frac{2v_iv_j + 2v_{i+1}v_{j+1} + v_iv_{j+1} + v_{i+1}v_j}{6}\delta_{ij}$$

so that L_p can be written as a quadratic function of the vertical displacements, by introducting the matrix $\mathbf{K}_p$:

$$L_p = ½v^TK_p^v \tag{25}$$

The matrix $\mathbf{K}_p$ is rather complex, and it is summarized in Table 1[1]. Finally, the strain energy L_p of the soil has to be expressed as a quadratic function of the Lagrangian coordinates, as follows:

$$L_p = ½c^TV^TK_pVc = ½c^T\tilde{K}_pc \tag{26}$$

Table 1. The Green soil matrix

$K_{p_{11}} = \frac{1}{3}r_{1,1}$	
$K_{p_{ii}} = \frac{1}{3}(r_{i,i} + r_{i-1,i-1} + r_{i,i-1})$	$i = 2, \ldots n-1$
$K_{p_{n,n}} = \frac{1}{3}r_{n-1,n-1}$	
$K_{p_{ij}} = \frac{1}{6}(2r_{ij} + r_{i,j-1} + r_{i-1,j} + 2r_{i-1,j-1})$	$1 < i < n-2$ $i < j < n$
$K_{p_{1j}} = \frac{1}{6}(2r_{1,j} + r_{1,j-1})$	$1 < j < n$
$K_{p_{in}} = \frac{1}{6}(r_{i,n-1} + 2r_{i-1,n-1})$	$1 < i < n-1$
$K_{p_{1,n}} = \frac{1}{6}r_{1,n-1}$	

The influence coefficients r_{ij} depend on the soil behaviour and therefore can be calculated if and only if the 'flexibility function' $c(z, z')$ is defined. In this way the Green displacement matrix **C** can be deduced, and its inverse **R** can be immediately calculated.

The total strain energy is given by:

$$L = \tfrac{1}{2}\mathbf{c}^T\mathbf{K}\mathbf{c} \tag{27}$$

where:

$$\mathbf{K} = \mathbf{K}_e + \mathbf{K}_f + \tilde{\mathbf{K}}_p. \tag{28}$$

2.3 The potential energy of the conservative axial forces

As mentioned previously, let us suppose that **P** is the n-dimensional array of the conservative forces.

The ith force P_i is applied at the ith cell absissa, and its potential energy is given by:

$$E_i = P_i w_i = \frac{1}{2} P_i \sum_{j=1}^{i-1} a\phi_i^2 \tag{29}$$

The potential energy of the n forces is equal to:

$$E = \frac{1}{2} \sum_{i=1}^{n} P_i \sum_{j=1}^{i-1} a\phi_i^2 = \frac{1}{2}\, \mathbf{c}^T\mathbf{B}\mathbf{c} \tag{30}$$

where **B** is a diagonal n-dimensional matrix, whose terms are given by:

$$B_{ii} = a \sum_{j=i+1}^{n} P_j \quad i = 1, \ldots n-1 \tag{31}$$

$$B_{n,n} = 0 \tag{32}$$

2.4. The virtual work of the applied follower forces

The virtual work of the non-conservative force Q_n can be calculated as follows. Let us assign an arbitrary vector of Lagrangian coordinates, and a virtual variation of the ith coordinates (Fig. 4). The virtual work of Q_n is given by:

$$\delta L_i = Q_n(\phi_i - \phi_{n-1})\delta\phi_i \tag{33}$$

in we assign a virtual rotation at the ith bar, and by:

$$\delta L_i = -Q_n \phi_{n-1} \delta_n \tag{34}$$

if a vertual displacement is assigned to the first cell.

The virtual work of Q_n can therefore be expressed as:

$$\delta L = S_n \delta c \tag{35}$$

where S_n is n-dimensional unsymmetrix matrix, whose elements are given by:

$$S_{ii}^{(n)} = Q_n \qquad i = 1, \ldots n-2 \tag{36}$$

$$S_{n,n}^{(n)} = 0 \tag{37}$$

$$S_{in-1}^{(n)} = -Q_n \qquad i = 1, \ldots n \tag{38}$$

If the force is acting at the kth cell, then the corresponding matrix will be given by:

$$S_{ii}^{(k)} = Q_k \qquad i = 1, \ldots k-1 \tag{39}$$

$$S_{i,k-1}^{(k)} = -Q_k \qquad i = 1, \ldots k \tag{40}$$

$$S_{n,k-1}^{(k)} = -Q_k \tag{41}$$

Finally, the matrix **S** of the nonconservative forces is given by:

$$\mathbf{S} = \sum_{k=2}^{n} \mathbf{S}^{(k)} \tag{42}$$

The Lagrange equations lead to the following equation of motion:

$$\tilde{\mathbf{M}}\ddot{\mathbf{c}} + (\mathbf{K} - \mu\mathbf{B} - \mu_1\mathbf{S})\mathbf{c} = 0 \tag{43}$$

and the following eigenvalue problem can be deduced:

$$(-\omega^2\tilde{\mathbf{M}} + \mathbf{K}_r)\mathbf{q} = 0. \tag{44}$$

It is is perhaps worth noting that a general eigenroutine must be used, because the matrix

$$\mathbf{K}_r = \mathbf{K} - \mu\mathbf{B} - \mu_1\mathbf{S} \tag{45}$$

is unsymmetric. We used a classical QR routine, as developed by Francis, or a subroutine from the 'DAMP' subroutine by Gupta.

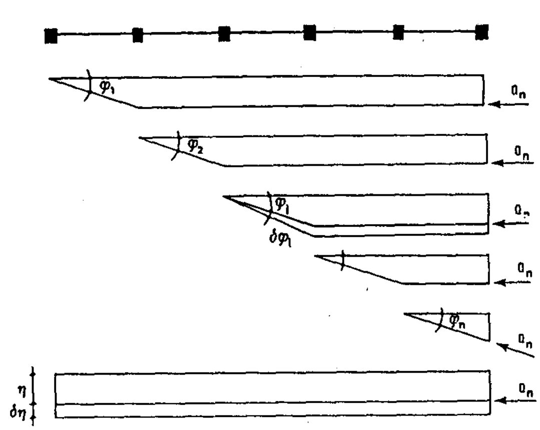

Fig. 4. Virtual variations of the Lagrangian coordinates

FREE VIBRATIONS IN THE ABSENCE OF AXIAL LOADS

If the structure is not subjected to external loadings, then the equations of motion reduce to:

$$\tilde{\mathbf{M}}\ddot{\mathbf{c}} + \mathbf{K}\mathbf{c} = \mathbf{0} \tag{46}$$

and the resulting eigenvalue problem has real eigenvalues. In fact, from a numerical point of view, this conservative problem is much simpler than the general one, and correspondingly simpler subroutines can be used.

As already stated, the soil model has to be defined, by choosing a flexibility function $c(z, z')$. While this choice is a complex problem, fortunately it is not of crucial importance here. We shall therefore assume a simple function, by hypothesing that the soil is homogeneous, but we would like to emphasize that the analysis remains unchanged if a more complex soil behaviour is assumed.

Let us define:

$$\zeta = \frac{z' - z}{1} \tag{47}$$

and let us assume that the proposed flexibility function has the following simple expression:

$$c(z, z') = c(\zeta) = \frac{c_0}{1 + rc_0} \tag{48}$$

The two parameters c_0 and r can be given a physical meaning, by observing that C_0 is equal to $c(0)$, and by imposing that r assumes a value c_0/m at a defined abscissa ζ_m. It follows that:

$$r = \frac{1}{c_0}\,\frac{m-1}{\zeta_m^2} \tag{49}$$

The real coefficient $m > 1$ can be called 'reduction factor' and the Winkler soil model is recovered as m goes to infinity (Fig. 5).

In order to examine the influence of the Green soil behaviour, let us suppose that the second moment of area and the distributed mass can be assumed to be constant along the span of the beam. It is also convenient to define the non dimensional 'soil coefficient':

$$\lambda = k_0 \frac{1^4}{EI} \tag{50}$$

where $k_0 = 1/c_0$, and the non dimensional frequency

$$C_i^4 = \left(\omega_i^2 \frac{EI}{1^4}\right) \tag{51}$$

In the Figs 6, 7 and 8 the first three adimensional frequencies are plotted vs. the soil coefficient λ, for various

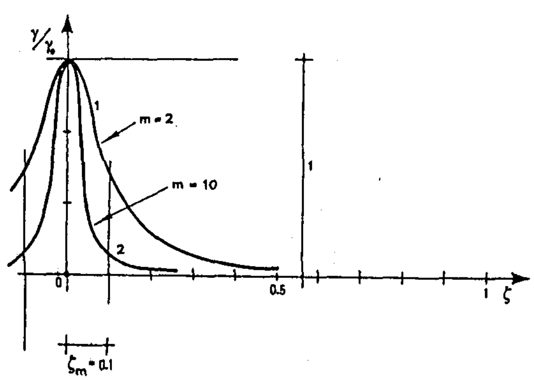

Fig. 5. The proposed flexibility function of the Green soil

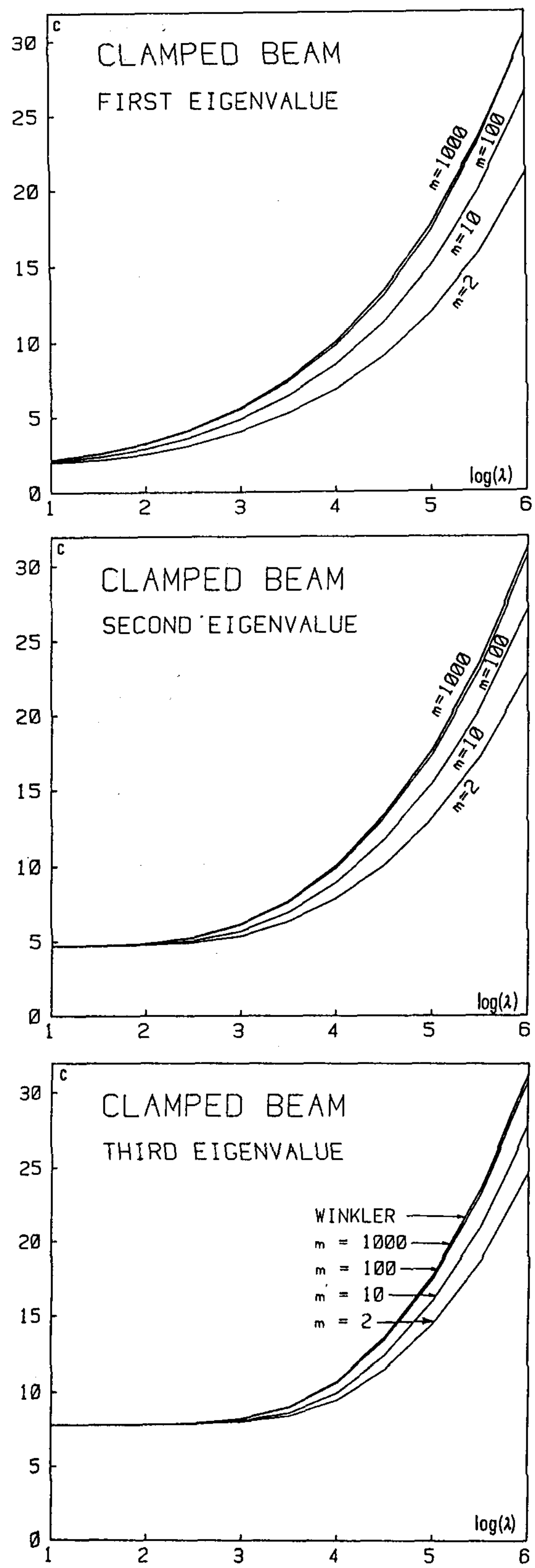

Fig. 6. Graph of the first three frequencies of the clamped-free beam vs. the nondimensional soil parameter

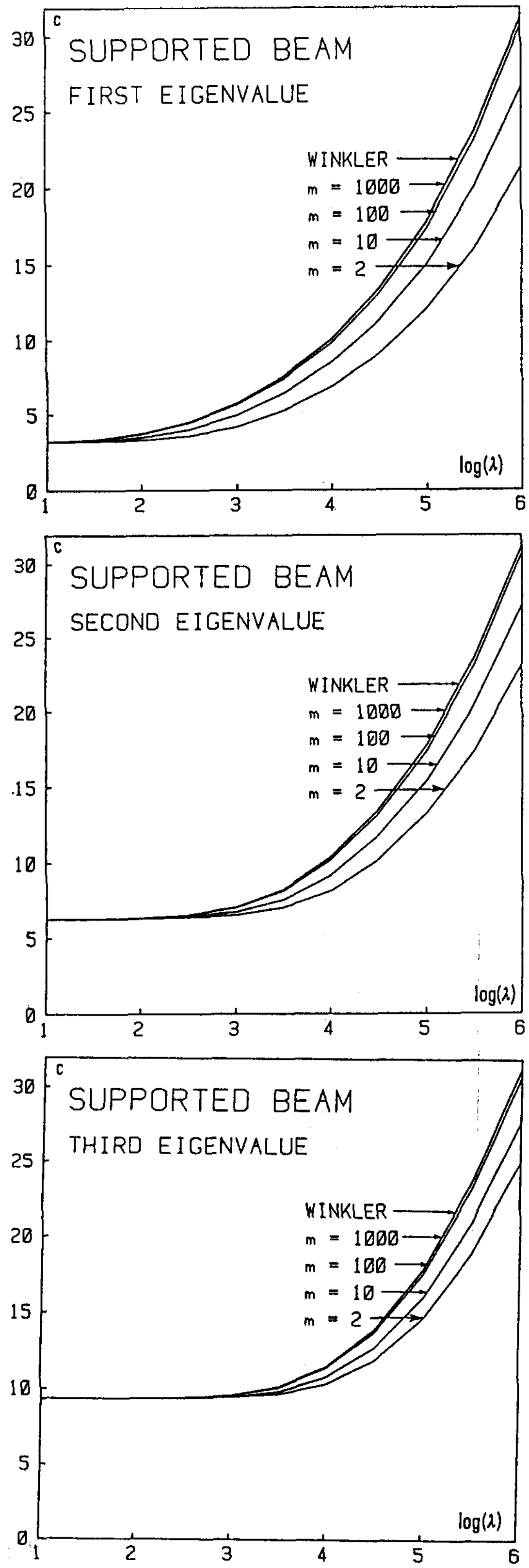

Fig. 7. *Graph of the first three frequencies of the simply supported beam vs. the nondimensional soil parameter*

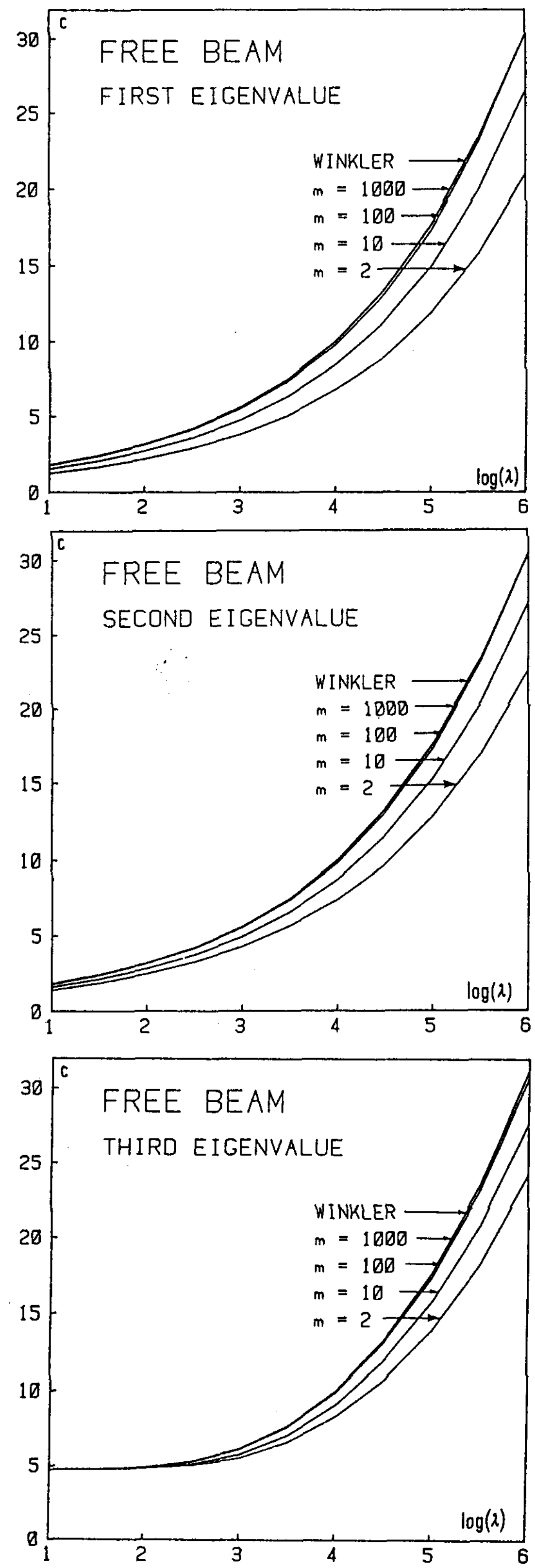

Fig. 8. *Graph of the first three frequencies of the free beam vs. the non dimensional soil coefficient*

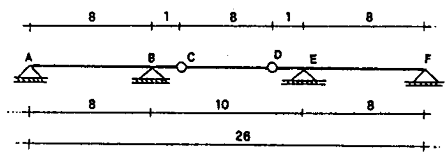

Fig. 9. A classical bridge structure

reduction factor values. Figure refers to the cantilever beam, Fig. 7 deals with the simply supported beam, whereas in Fig. 8 the free beam is examined.

In each case the reduction factor $m = 1000$ reproduces the Winkler values, while lower reduction factor values lead to noticeable lowering of the frequencies, at least if λ is large enough.

Another, more complex beam is sketched in Fig. 9. It is a typical bridge structure, whose total span is equal to 26 meters, while the two lateral spans are 10 meters each. The inner hinges are symmetrically placed at $z = 10$ and $z = 16$ meters. The beam has been studied by dividing it into 26 rigid bars, and in Fig. 10 the graph of the first two adimensional frequencies is sketched vs. non dimensional soil coefficient, for various reduction factor values.

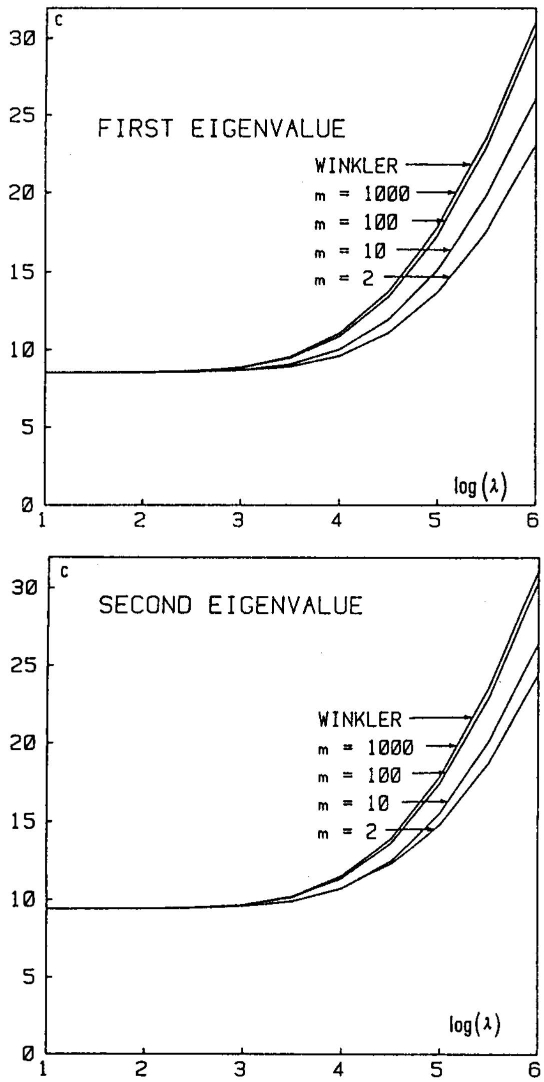

Fig. 10. Graph of the first two frequencies of the beam in Fig. 9 vs the non dimensional soil coefficient

STABILITY ANALYSIS IN THE PRESENCE OF CONSERVATIVE AXIAL LOADS

If conservative axial forces act on the system, then the equation of motion (43) reduces to:

$$\tilde{\mathbf{M}}\ddot{\mathbf{c}} + (\mathbf{K} - \mu\mathbf{B})\mathbf{c} = 0 \tag{52}$$

and the corresponding eigenproblem:

$$[-\omega^2\tilde{\mathbf{M}} + (\mathbf{K} - \mu\mathbf{B})]\mathbf{q} = 0 \tag{53}$$

has real eigensolutions. As the axial loads increase, the frequencies decrease, and the first frequency becomes zero when μ reaches the critical value. In this case, instability can only occur by a divergence mechanism, so that the critical load values can be detected more directly by imposing:

$$\det(\mathbf{K} - \mu\mathbf{B}) = 0 \tag{54}$$

The emphasis is here on the soil influence, so that we shall examine the critical load as a function of the two soil parameters λ and m.

In Figs 11 and 12 the graph of the non-dimensional critical load:

$$\gamma = \frac{Fl^2}{EI} \tag{55}$$

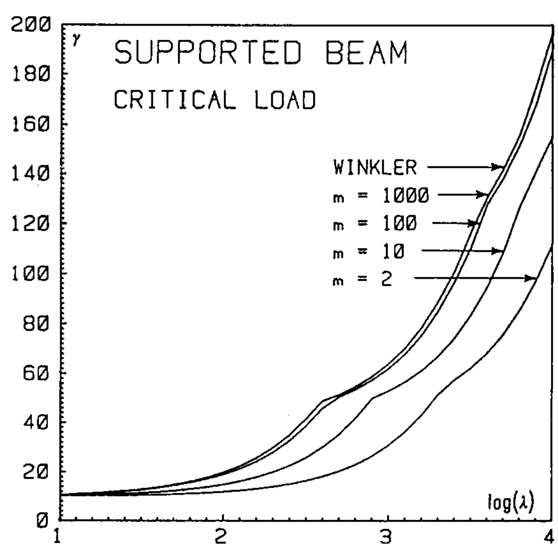

Fig. 11 Graph of the critical load of a simply supported beam subjected to a conservative axial load at its right end, vs. the non dimensional soil parameter

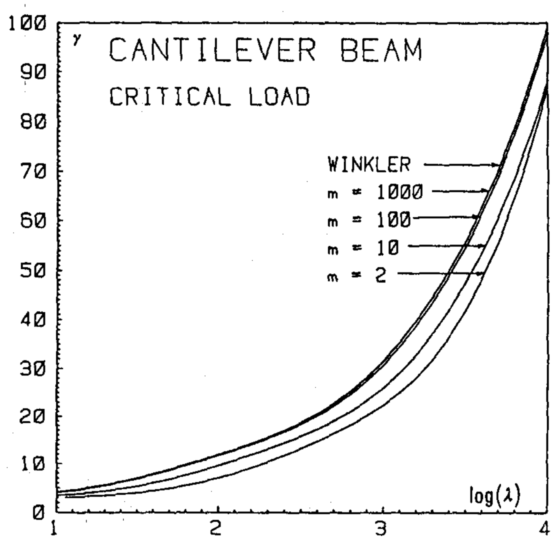

Fig. 12. Graph of the critical load of a cantilever beam subjected to a conservative axial load at its right end, vs the non dimensional soil parameter

is reported as a function of the non dimensional soil parameter, for various values of the reduction factor m. Figure 11 deals with the simply supported beam subjected to a single axial force F at the right end, while the cantilever beam is examined in figure 12 for the same loading condition. In figure 11 the points of overlapping critical loads are clearly shown, corresponding to a well-known phenomenon for the simply supported beam on Winkler soil (see for example Ref. 3 pp. 50–54).

INSTABILITY MECHANISM OF PSEUDO-CONSERVATIVE SYSTEMS ON ELASTIC SOIL

The well known Sundararajan theorem[4] states that 'the critical load of an undamped, linearly elastic column subjected to stationary forces (conservative or nonconservative) does not decrease due to the introduction of a Winkler type elastic foundation having modulus distribution geometrically similar to the mass distribution of the column'.

Moreover, a number of other papers have shown that the flutter load of the Beck rod and Pflüger rod is unaffected by the presence of the soil[15], while in other cases the flutter load can decrease with dissimilar distribution of the column mass and foundation stiffness[6]. Therefore, it seems useless to examine the influence of the soil on the flutter loads of non-conservaative systems.

On the other hand, a class of structures exist which are subjected to non-conservative forces, but nevertheless exhibit divergence critical loads. These kind of structures are called pseudo-divergence systems by Leipholz[7] or also pseudo-conservative systems by Huseyin[8], and are mathematically recognized because the product

$$\tilde{\mathbf{M}}^{-1}\mathbf{K}_r$$

is symmetrizable[8]. From a physical point of view, they always lack stability by divergence, but additional flutter loads exist, whose magnitude is greater than the divergence load. Thereofre, Leipholz states that 'these flutter loads are of little practical importance' (Ref. p.331).

In the following, we shall show that these flutter loads regain their importance, because the soil presence can transform the pseudo-divergence systems in truly nonconservative systems, whose primary flutter load coincides with the previously defined flutter load. To be more precise, let us consider a clamped-clamped beam with constant cross section and constant mass distribution. If the beam is subjected to uniformaly distributed follower forces (Fig. 13), then it is known that the first two critical loads are given by:

$$g_{crit}^{(1)} = 80.26 \frac{EI}{l^3}$$

$$g_{crit}^{(2)} = 139.36 \frac{EI}{l^3}$$

In addition, a flutter load g_{fl} exists, whose magnitude is greater than $g_{crit}^{(2)}$.

If the soil influence is considered, then the flutter load remains unaffected, while the divergence loads increase. A (critical) value of the soil parameter will exist, which corresponds to passage from divergence to flutter. Mathematically, this means that the product:

$$\tilde{\mathbf{M}}^{-1}\mathbf{K}_r$$

is no more symmetrizable.

In Fig. 14 the behaviour of the first two frequencies is plotted vs. the adimensional load parameter:

$$\gamma = \frac{gl^3}{EI}$$

for various values of the modulus of subgrade regions, and by hypothesing Winkler soil.

As it is immediate to realize, the system behaviour is dramatically changed by the soil influence.

In fact this is quite a general conclusion: the whole graph is shifted to the 'right' by the soil presence, so that every pseudo-conservative system will become a nonconservative system, if a sufficiently strong soil is added.

CONCLUSIONS

The free vibrations of a foundation beam on Green-Boussinesqu soil have been calculated, in the presence of

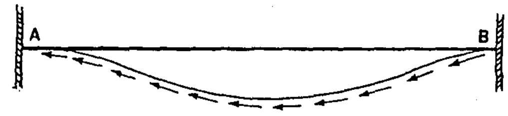

Fig. 13. Clamped-clamped beam subjected to uniformly distributed follower forces: one of the simplest example of pseudo-divergence system

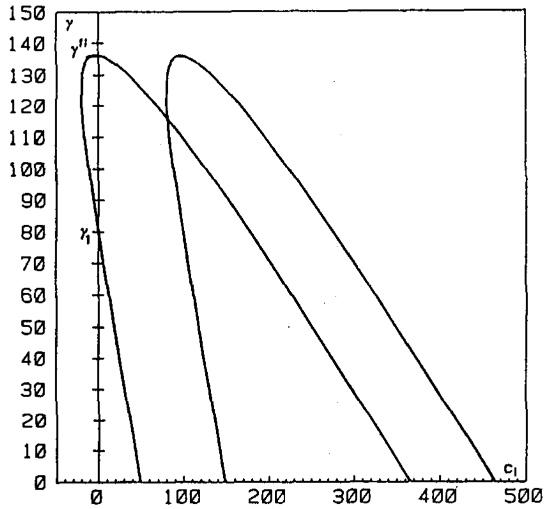

Fig. 14. Graph of the first two frequencies of the beam in Fig. 13, vs. the follower load parameter

conservative and nonconservative axial loads. The influence of the soil behaviour is examined, with respect to the frequencies of free vibrations in the absence of applied loads, and non dimensional graphs are given for three common beams and a more complex structure. From these graphs it is possible to deduce:

(i) the frequencies of a foundation beam on Green soil are always lower than the corresponding frequencies of the beam on Winkler soil,

(ii) the differences are significant only if the soil is quite rigid.

The critical load of cantilever beams and simply supported beams is plotted as functions of the two soil parameters. It is evident from these graphs that the critical load of a foundation beam on Green soil is lower than the corresponding critical load of the beam on Winkler soil. Finally, pseudo-divergence systems are shown to become truly non-conservative systems if sufficiently rigid soil is introduced.

REFERENCES

1 Franciosi, V. and Franciosi, C. Foundation beams on Green soil: an interesting use of the Cells method, *Studi e Ricerche*, **10**, Politecnico di Milano, Italy

2 De Rosa, M. A. Stability and dynamics of beams on Winkler elastic foundation, to be published on Earth. Engng. Struct. Dyn.

3 Thompson, J. M. T. *Instabilities and Catastrophies in Science and Engineering*, J. Wiley, Chichester, 1982

4 Sundararajan, C. Stability of columns on elastic foundations subjected to conservative or non-conservative forces, *J. Sound Vibr.* 1974, **37**, 79–85

5 Jacoby, A. and Elishakoff, I. Discrete-continuous elastic foundation may leave the flutter load of Pflüger column unaffected, *J. Sound Vibr.*, 1986, **108**(3), 535–525

6 Hauger, W. and Vetter, K. Influence of an elastic foundation on the stability of a tangentially loaded column, *J. Sound Vibr.*, 1976, **47**, 296–299

7 Leipholz, H. *Stability of Elastic Systems*, Sijjthoff & Noordhoff Alphen aan den Rjin, The Netherlands, 1980

8 Huseyin, R. *Vibrations and Stability of Multiple Parameter Systems* Noordhoff, Alphen aan den Rjin, The Netherlands, 1978