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LETTERS TO THE EDITOR

FREE VIBRATIONS OF ARCHES IN PRESENCE OF AXIAL FORCES

1. INTRODUCTION

Free vibrations of circular and parabolic arches have been already studied in detail, both for clamped arches and for two- and three-hinged arches [1–5]. In each case it is possible to define *ad hoc* non-dimensional coefficients, which allow the tabulation of the frequencies. In this letter the arch is reduced to a set of rigid bars and lumped masses, linked together by means of elastic "cells" [5, 6], and the destabilizing effect of the dead load is investigated. A new non-dimensional coefficient is proposed, and the frequencies of free vibration are tabulated for clamped and two- and three-hinged arches. Finally, two examples show how to use the tables, for deep arches and shallow arches respectively.

2. THE EQUATION OF MOTION

Consider the arch shown in Figure 1. In order to calculate the frequencies of free vibration, together with the associated vibrational modes, it is helpful to reduce the structure to a finite n-degree-of-freedom system. It seems that the so-called "cell procedure" can be successfully applied to one-dimensional structures, leading to simpler

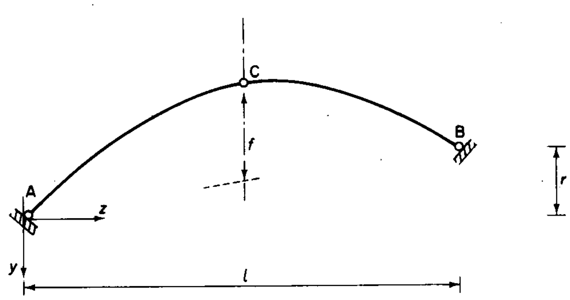

Figure 1. Three-hinged arch model.

models than in FEM reductions [6]. According to this method, the arch is divided into t rigid bars, connected by means of elastic "cells", in which the strain energy of the bars is concentrated. The masses are lumped accordingly. The structure is reduced to a $t-2$ degree of freedom system, and the rotations of the first $t-2$ bars can be assumed as Lagrangian co-ordinates. Of course, displacements also can be chosen as Lagrangian co-ordinates, but it seems that the choice being made here leads to better estimates of the internal stresses.

The differential equation of motion can be written as

$$M\ddot{\phi}+(K-\mu B)\phi=0, \tag{1}$$

where M is the Lagrangian mass matrix, K is the stiffness matrix, B is the reduced matrix of the axial forces and $\phi(t)$ is the vector of the Lagrangian co-ordinates. An effective way to obtain these matrices was presented in reference [6].

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TABLE 1

Frequency coefficients λ for the first six modes of clamped arches; all the calculations were performed with n = 20 Lagrangian co-ordinates

f/l \ χ	0	10	20	30	40	50	60	70	80	90	100
0·05	7·711	7·150	6·401	5·198							
	10·31	9·872	9·356	8·734							
	13·71	13·38	13·01	12·61							
	16·36	16·07	15·76	15·44							
	19·41	19·16	18·90	18·63							
	21·94	21·72	21·49	21·25							
0·10	7·517	7·247	6·941	6·587	6·160	5·614	4·827	3·065			
	10·17	9·951	9·715	9·460	9·181	8·873	8·527	8·133			
	13·47	13·30	13·12	12·93	12·73	12·53	12·32	12·09			
	16·13	15·98	15·83	15·67	15·51	15.34	15.17	14.99			
	19·09	18·96	18·83	18·70	18·56	18·42	18·28	18·14			
	21·62	21·51	21·39	21·27	21·15	21·03	20·90	20·78			
0·15	7·228	7·046	6·847	6·628	6·384	6·106	5·783	5·392	4·887	4·138	1·944
	9·948	9·797	9·637	9·469	9·291	9·102	8·900	8·683	8·448	8·191	7·906
	13·11	12·99	12·86	12·74	12·60	12·47	12·33	12·18	12·03	11·87	11·71
	15·77	15·66	15·55	15·45	15·34	15·22	15·10	14·99	14·86	14·74	14·61
	18·62	18·53	18·44	18·34	18·25	18·15	18·06	17·96	17·86	17·75	17·65
	21·13	21·05	20·97	20·88	20·80	20·71	20·63	20·54	20·45	20·36	20·27
0·20	6·886	6·743	6·590	6·425	6·245	6·048	5·829	5·580	5·292	4·945	4·503
	9·665	9·544	9·419	9·288	9·152	9·010	8·858	8·699	8·530	8·351	8·159
	12·68	12·59	12·49	12·38	12·28	12·17	12·06	11·95	11·83	11·71	11·58
	15·31	15·23	15·14	15·05	14·97	14·87	14·78	14·69	14·59	14·49	14·39
	18·05	17·97	17·90	17·82	17·75	17·67	17·59	17·15	17·43	17·35	17·27
	20·52	20·45	20·38	20·31	20·25	20·18	20·11	20·04	19·96	19·89	19·82
0·25	6·522	6·400	6·271	6·133	5·984	5·823	5·647	5·452	5·233	4·980	4·682
	9·340	9·236	9·129	9·017	8·901	8·780	8·654	8·522	8·383	8·237	8·083
	12·22	12·14	12·05	11·96	11·87	11·78	11·68	11·58	11·48	11·38	11·27
	14·80	14·73	14·65	14·57	14·49	14·42	14·34	14·25	14·17	14·09	14·00
	17·42	17·35	17·29	17·22	17·15	17·09	17·02	16·95	16·88	16·80	16·73
	19·82	19·76	19·70	19·64	19·58	19·52	19·46	19·40	19·34	19·27	19·21
0·3	6·161	6·052	5·935	5·811	5·679	5·536	5·380	5·209	5·020	4·805	4·556
	8·993	8·899	8·801	8·699	8·594	8·485	8·372	8·253	8·129	7·999	7·862
	11·76	11·68	11·60	11·51	11·43	11·34	11·25	11·16	11·07	10·98	10·88
	14·26	14·19	14·12	14·05	13·98	13·90	13·83	13·75	13·67	13·60	13·52
	16·16	16·70	16·64	16·58	16·52	16·45	16·39	16·32	16·26	16·19	16·12
	19·09	19·04	18·98	18·93	18·87	18·81	18·75	18·69	18·64	18·58	18·52
0·35	5·817	5·714	5·605	5·488	5·363	5·229	5·083	4·923	4·745	4·544	4·311
	8·638	8·549	8·456	8·360	8·261	8·157	8·050	7·938	7·820	7·680	7·568
	11·29	11·21	11·14	11·06	10·98	10·89	10·81	10·72	10·63	10·54	10·45
	13·71	13·64	13·57	13·5	13·44	13·36	13·29	13·22	13·14	13·07	12·99
	16·11	16·05	15·99	15·93	15·87	15·80	15·74	15·68	15·61	15·55	15·48
	18·36	18·30	18·25	18·19	18·14	18·08	18·02	17·97	17·91	17·85	17·79
0·4	5·497	5·397	5·290	5·177	5·055	4·923	4·780	4·622	4·445	4·244	4·008
	8·287	8·200	8·109	8·015	7·918	7·816	7·711	7·601	7·486	7·365	7·237
	10·84	10·76	10·69	10·61	10·53	10·44	10·36	10·27	10·18	10·09	9·999
	13·17	13·10	13·03	12·96	12·89	12·82	12·75	12·68	12·60	12·53	12·45
	15·46	15·40	15·34	15·28	15·22	15·15	15·09	15·03	14·97	14·90	14·83
	17·63	17·57	17·52	17·46	17·41	17·35	17·29	17·24	17·18	17·42	17·06

TABLE 1 (*cont.*)

f/l \ χ	0	10	20	30	40	50	60	70	80	90	100
0·45	5·203	5·103	4·997	4·883	4·761	4·628	4·482	4·319	4·136	3·923	3·668
	7·947	7·860	7·769	7·675	7·577	7·476	7·369	7·258	7·141	7·018	6·888
	10·40	10·32	10·25	10·17	10·09	10·00	9·917	9·829	9·739	9·645	9·549
	12·64	12·57	12·50	12·44	12·36	12·29	12·22	12·14	12·07	11·99	11·91
	14·84	14·78	14·72	14·65	14·59	14·53	14·46	14·40	14·33	14·26	14·20
	16·92	16·87	16·81	16·76	16·70	16·64	16·58	16·52	16·46	16·40	16·34
0·5	4·933	4·833	4·725	4·609	4·484	4·346	4·183	4·021	3·823	3·586	3·290
	7·622	7·531	7·442	7·346	7·246	7·142	7·033	6·918	6·797	6·669	6·532
	9·983	9·906	9·826	9·745	9·661	9·575	9·487	9·396	9·302	9·205	9·104
	12·14	12·07	12·00	11·93	11·85	11·77	11·70	11·62	11·54	11·46	11·38
	14·24	14·18	14·12	14·05	13·99	13·92	13·86	13·79	13·72	13·65	13·58
	16·25	16·20	16·14	16·08	16·02	15·96	15·90	15·84	15·77	15·71	15·65

TABLE 2

Frequency coefficients λ *for the first six modes of two-hinged arches; all the calculations were performed with* $n = 20$ *Lagrangian co-ordinates*

f/l \ χ	0	5	10	15	20	25	30	35	40	45
0·05	6·199	5·727	5·098	4·072						
	9·071	8·779	8·455	8·089						
	12·320	12·101	11·870	11·624						
	15·176	15·002	14·822	14·636						
	18·179	18·033	17·882	17·729						
	20·905	20·779	20·650	20·519						
0·10	6·023	5·794	5·532	5·228	4·858	4·377	3·648			
	8·935	8·787	8·631	8·466	8·290	8·103	7·902			
	12·108	11·995	11·880	11·760	11·637	11·510	11·379			
	14·956	14·866	14·774	14·681	14·586	14·489	14·390			
	17·898	17·822	17·745	17·667	17·588	17·508	17·427			
	20·604	20·539	20·473	20·406	20·338	20·270	20·201			
0·15	5·770	5·611	5·436	5·243	5·025	4·775	4·478	4·106	3·592	2·641
	8·723	8·619	8·510	8·397	8·280	8·157	8·028	7·894	7·751	7·601
	11·797	11·72	11·635	11·552	11·467	11·380	11·291	11·200	11·106	11·010
	14·617	14·55	14·488	14·423	14·356	14·289	14·220	14·151	14·080	14·008
	17·476	17·42	17·367	17·313	17·257	17·201	17·144	17·087	17·029	16·970
	20·143	20·10	20·049	20·002	19·954	19·906	19·858	19·809	19·760	19·710
0·20	5·476	5·347	5·208	5·057	4·891	4·706	4·497	4·253	3·957	3·576
	8·455	8·370	8·282	8·191	8·097	8·000	7·899	7·793	7·684	7·569
	11·424	11·359	11·292	11·225	11·156	11·086	11·014	10·941	10·867	10·791
	14·194	14·141	14·088	14·034	13·980	13·925	13·869	13·813	13·756	13·698
	16·859	16·914	16·870	16·825	16·779	16·733	16·687	16·641	16·593	16·546
	19·567	19·528	19·489	19·450	19·411	19·372	19·332	19·293	19·252	19·212
0·25	5·170	5·056	4·935	4·803	4·660	4·502	4·325	4·123	3·886	3·595
	8·151	8·075	7·997	7·917	7·835	7·749	7·660	7·569	7·473	7·375
	11·02	10·961	10·902	10·842	10·780	10·719	10·656	10·592	10·526	10·459
	13·717	13·671	13·623	13·576	13·527	13·479	13·429	13·379	13·329	13·278
	16·385	16·346	16·306	16·266	16·225	16·184	16·143	16·102	16·060	16·018
	18·919	18·884	18·850	18·815	18·780	18·745	18·710	18·674	18·639	18·602

TABLE 2 (*cont.*)

f/l \ χ	0	5	10	15	20	25	30	35	40	45
0·3	4·871	4·765	4·652	4·529	4·395	4·248	4·083	3·895	3·675	3·405
	7·828	7·757	7·685	7·609	7·532	7·452	7·369	7·283	7·194	7·101
	10·599	10·545	10·489	10·433	10·376	10·318	10·258	10·198	10·137	10·075
	13·215	13·171	13·126	13·081	13·036	12·990	12·994	12·897	12·849	12·802
	15·784	15·747	15·709	15·672	15·633	15·595	15·557	15·518	15·478	15·439
	18·235	18·203	18·171	18·137	18·105	18·071	18·038	18·005	17·971	17·937
0·35	4·590	4·487	4·377	4·256	4·125	3·979	3·815	3·626	3·401	3·118
	7·502	7·433	7·312	7·288	7·212	7·134	7·053	6·968	6·881	6·790
	10·178	10·126	10·072	10·017	9·961	9·904	9·846	9·787	9·727	9·666
	12·705	12·662	12·619	12·575	12·531	12·486	12·440	12·395	12·348	12·301
	15·177	15·141	15·104	15·068	15·030	14·993	14·995	14·917	14·879	14·840
	17·542	17·511	17·479	17·447	17·415	17·383	17·350	17·318	17·285	17·252
0·40	4·331	4·228	4·117	3·995	3·861	3·710	3·358	3·336	3·088	2·757
	7·183	7·113	7·042	6·968	6·890	6·812	6·730	6·644	6·554	6·461
	9·766	9·713	9·659	9·604	9·548	9·491	9·433	9·373	9·313	9·251
	12·202	12·159	12·116	12·072	12·028	11·983	11·936	11·890	11·844	11·797
	14·580	14·544	14·507	14·470	14·433	14·395	14·357	14·319	14·280	14·282
	16·860	16·828	16·796	16·764	16·732	16·700	16·667	16·634	16·601	16·568
0·45	4·094	3·989	3·875	3·748	3·608	3·447	3·260	3·032	2·734	2·283
	6·876	6·805	6·732	6·656	6·577	6·495	6·410	6·32	6·227	6·129
	9·368	9·314	9·259	9·202	9·144	9·086	9·026	8·96	8·902	8·839
	11·716	11·672	11·628	11·583	11·537	11·491	11·444	11·397	11·349	11·300
	14·002	13·965	13·928	13·890	13·851	13·813	13·774	13·735	13·696	13·656
	16·199	16·166	16·133	16·101	16·069	16·035	16·001	15·97	15·934	15·899
0·5	3·8790	3·770	3·651	3·5177	3·366	3·191	2·978	2·705	2·306	1·292
	6·585	6·512	6·435	6·357	6·275	6·189	6·099	6·005	5·905	6·800
	8·988	8·932	8·875	8·816	8·756	8·695	8·633	8·569	8·503	8·436
	11·250	11·205	11·159	11·112	11·065	11·018	10·969	10·920	10·870	10·819
	13·449	13·411	13·372	13·333	13·293	13·254	13·213	13·173	13·132	13·09
	15·566	15·53	15·499	15·465	15·431	15·396	15·361	15·326	15·291	15·256

If the matrix B is not considered ($\mu = 0$), then equation (1) is valid for every arch axis and every arch cross section. If on the contrary the destabilizing effects of the matrix B should be introduced, then the hypothesis must be made that the arch axis is the thrust line of the load. For example, a parabolic arch corresponds to an uniformly distributed load. This hypothesis is by no means restrictive, if arch bridges are being considered, in which the dead load is by far the most important load on the structure.

Consider now the arch shown in Figure 1, where the abutments may not be at the same level. The span is l, the rise is f, and the area and the inertia of the cross section vary according to the generic laws $A(z) = A_0 f(z)$ and $I(z) = I_0 g(z)$. The dead load can be supposed equal to $q(z) = A(z) = q_0 f(z)$. ρ is the mass density of the material and g is the acceleration of gravity.

It is possible to introduce the non-dimensional parameters $\chi^2 = q_0^2/\rho^2 I_0$ and $\lambda^4 = q_0 \omega^2 l^4 / gEI_0$, and then to tabulate the λ values as functions of χ and of f/l. It is worth noting that the parameter λ is equal to the Sakiyama parameter [4]; here the shear effects are being neglected and Sakiyama's second parameter a is unnecessary.

In Tables 1-3 the first six λ values are given for parabolic arches; these correspond to the first six eigenvalues. In some cases instability was reached, and no values are reported.

Table 3 concerns three-hinged arches; hence the Sakiyama values are also given, for the sake of comparison. As it is possible to see, the agreement is always quite good, if the destabilizing effects are neglected. It is worth noting that some discrepancies appear for steep arches, because here it is supposed that the dead load is uniformly distributed along the horizontal. This is necessary if the arch must be the thrust line of the load.

It is interesting to observe that the curve $\omega^2 - q$ *shows its convexity towards the origin,* as predicted by Huseyin [7].

TABLE 3

Frequency coefficients λ *for the first six modes of three-hinged arches; all the calculations were performed with* $n = 20$ *Lagrangian co-ordinates; the first column gives the Sakiyama values* [4]

f/l \ χ	Sakiyama [4]	0	5	10	15	20	25	30
0·05	6·209	6·199	5·7273	5·098				
	7·432	7·428	6·681	5·487				
	12·108	12·32	12·101	11·870				
	12·406	13·59	13·319	13·026				
	14·198	18·179	18·032	17·883				
	18·672	19·337	19·166	18·989				
0·10	6·009	6·0238	5·794	5·532	5·228	4·858	3·950	
	7·354	7·3269	6·978	6·559	6·033	5·299	4·377	
	12·184	12·108	11·995	11·88	11·76	11·637	11·51	
	13·552	13·401	13·261	13·116	12·966	12·812	12·651	
	17·027	17·898	17·822	17·745	17·667	17·588	17·508	
	18·296	19·065	18·976	18·887	18·796	18·704	18·611	
0·15	5·179	5·7704	5·611	5·436	5·243	5·025	4·775	4·478
	7·167	7·168	6·933	6·668	6·365	6·008	5·566	4·977
	11·784	11·797	11·717	11·635	11·553	11·467	11·380	11·291
	13·213	13·105	13·007	12·91	12·803	12·698	12·589	12·479
	17·732	17·476	17·422	17·37	17·313	17·257	17·201	17·144
	18·973	18·647	18·584	18·521	18·458	18·394	18·329	18·263
0·2	5·382	5·4759	5·347	5·208	5·057	4·891	4·7066	4·497
	6·916	6·964	6·781	6·580	6·356	6·107	5·820	5·480
	11·303	11·424	11·359	11·293	11·225	11·156	11·086	11·014
	12·728	12·734	12·655	12·574	12·492	12·408	12·323	12·236
	17·043	16·959	16·915	16·870	16·825	16·780	16·734	16·688
	18·424	18·124	18·074	18·023	17·971	17·919	17·876	17·814
0·25	5·033	5·1699	5·056	4·9348	4·8033	4·660	4·502	4·325
	6·619	6·7282	6·573	6·405	6·222	6·0207	5·796	5·539
	10·777	11·019	10·961	10·902	10·842	10·781	10·719	10·656
	12·173	12·317	12·248	12·178	12·107	12·034	11·960	11·885
	16·281	16·385	16·346	16·306	16·266	16·225	16·184	16·143
	17·644	17·537	17·492	17·447	17·402	17·357	17·311	17·265
0·3	4·692	4·8714	4·765	4.652	4·529	4·395	4·248	4·083
	6·293	6·4744	6·335	6·186	6·024	5·848	5·654	5·436
	10·232	10·599	10·545	10·489	10·433	10·376	10·313	10·259
	11·590	11·877	11·814	11·750	11·684	11·618	11·550	11·481
	15·490	15·784	15·747	15·709	15·672	15·634	15·595	15·56
	16·814	16·917	16·876	16·834	16·792	16·750	16·707	16·664

TABLE 3 (*cont.*)

f/l \ χ	Sakiyama [4]	0	5	10	15	20	25	30
0·35		4·5903	4·487	4·3765	4·2564	4·125	3·979	3·8148
		6·2135	6·0836	5·9445	5·7946	5·632	5·453	5·2544
		10·178	10·125	10·071	10·017	9·961	9·904	9·846
		11·432	11·3718	11·3106	11·248	11·185	11·120	11·055
		15·177	15·141	15·104	15·067	15·03	14·993	14·955
		16·2897	16·25	16·21	16·17	16·129	16·088	16·047
0·4	4·078	4·331	4·228	4·1165	3·995	3·861	3·7103	3·538
	5·620	5·954	5·8288	5·695	5·551	5·394	5·2225	5·0315
	9·160	9·766	9·7131	9·659	9·604	9·548	9·491	9·433
	10·431	10·993	10·934	10·874	10·813	10·751	10·687	10·623
	13·928	14·58	14·544	14·507	14·470	14·433	14·395	14·357
	15·169	15·671	15·632	15·592	15·553	15·513	15·472	15·431
0·45		4·0941	3·989	3·8746	3·7485	3·6076	3·4472	3·2597
		5·7014	5·578	5·4463	5·3037	5·1486	4·9777	4·787
		9·368	9·314	6·2586	9·2022	9·1448	9·086	9·026
		10·556	10·507	10·447	10·386	10·3243	10·261	10·197
		14·002	13·965	13·9276	13·889	13·852	13·813	13·774
		15·072	15·032	14·993	14·953	14·913	14·87	14·831
0·5	3·568	3·8788	3·7704	3·651	3·5177	3·366	3·1907	2·978
	4·987	5·4595	5·336	5·024	5·0599	4·902	4·728	4·532
	8·179	8·988	8·932	8·875	8·8161	8·756	8·695	8·633
	9·362	10·156	10·097	10·037	9·9746	9·912	9·847	9·782
	12·495	13·449	13·411	13·372	13·333	13·294	13·253	13·213
	13·659	14·498	14·458	14·417	14·376	14·335	14·294	14·252

3. NUMERICAL EXAMPLES

For the parabolic arch example shown in Figure 2, the span is 200 m, and the rise is 50 m. The dead load is supposed to be constant, $q = 225\ \text{t m}^{-1}$, together with the moment of inertia $I = 29{\cdot}97\ \text{m}^4$. In Table 4 the values of the first three periods are reported. In the first column the Sakiyama theory was used, while second and third columns give the period values in the absence and the presence of axial loads, respectively. The fourth column contains the first three periods when the masses are distributed along the arch axis, and not along the horizontal.

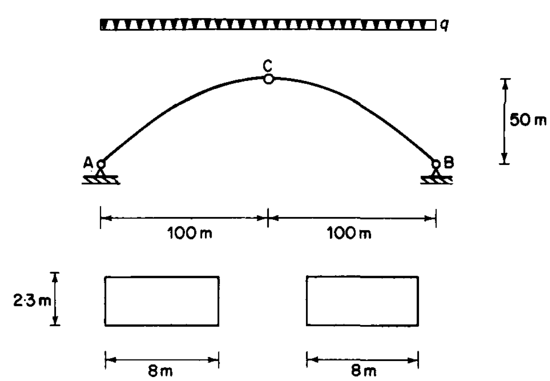

Figure 2. A steep three-hinged arch.

TABLE 4

First three periods for the steep arch shown in Figure 2; *the first column refers to Sakiyama values* [4]

Sakiyama [4]	$\chi=0$	$\chi=16{\cdot}44$	$\chi=0$
5·011	4·749	5·598	5·102
2·6539	2·804	3·341	2·954
1·0929	1·045	1·083	1·127

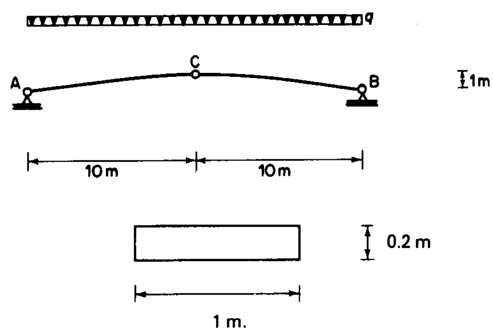

Figure 3. A shallow three-hinged arch.

TABLE 5

First six periods for the shallow arch shown in Figure 3; *the first column refers to Sakiyama values* [4]

Sakiyama [4]	$\chi=0$	$\chi=7{\cdot}6$
0·326	0·327	0·429
0·227	0·228	0·339
0·086	0·083	0·087
0·081	0·068	0·072
0·062	0·038	0·039
0·036	0·034	0·035

A second example is the shallow parabolic arch shown in Figure 3, whose span is 20 m, while the rise is 1 m. Accordingly, the load is uniformly distributed, and equal to $0{\cdot}49\ \mathrm{t\ m^{-1}}$, while the moment of inertia is equal to $6{\cdot}666\times10^{-4}\ \mathrm{m}^4$. In Table 5 the values of the first six periods are shown, from which it is possible to realize the effective importance of the destabilizing effects of the dead load.

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REFERENCES

1. A. S. VELETSOS, W. J. AUSTIN, C. A. PEREIRA and S. WUNG 1972 *Proceedings of the American Society of Civil Engineers* **98**, 311-329. Free-in-plane vibration of circular arches.
2. W. J. AUSTIN and A. S. VELETSOS 1973 *Proceedings of the American Society of Civil Engineers* **99**, 735-753. Free vibration of arches flexible in shear.
3. T. IRIE, G. YAMADA and K. TANAKA 1983 *Journal of Applied Mechanics, American Society of Mechanical Engineers* **50**, 449-452. Natural frequencies of in-plane vibration of arches.
4. T. SAKIYAMA 1986 *Journal of Sound and Vibration* **111**, 343-348. Free vibration of three-hinged arches.
5. A. RAITHEL and C. FRANCIOSI 1984 *Journal of Structural Engineering, American Society of Civil Engineers* **110**, 847-858. Dynamic analysis of arches using Lagrangian approach.
6. A. RAITHEL and C. FRANCIOSI 1985 *International Journal of Solids and Structures* **21**, 427-446. The stability of arches in the Lagrangian approach.
7. K. HUSEYIN 1978 *Vibration and Stability of Multiple Parameter Systems*. Alphen Aan Den Rijn: Noordhoff.