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DYNAMIC ANALYSIS OF NON-UNIFORM THIN WALLED BEAMS WITH NON-CLASSICAL BOUNDARY CONDITIONS

C. Franciosi and Zhong-Xian Li†

Department of Structural Engineering and Soil Mechanics, Facoltà di Ingegneria, Università della Basilicata, Potenza 85100, Italy

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Abstract—A simple numerical approach to the dynamic behaviour of a general thin-walled beam is proposed, in the presence of flexible constraints and non-uniform cross-section. Coupled bending and torsional vibration frequencies can be obtained for an arbitrary non-symmetrical cross-section, and the presence of intermediate concentrated masses and elastic constraints can be easily dealt with. The equations of motion are obtained by using the Lagrange equation, and explicit formulae for the strain energy and the kinetic energy are given. The highly banded form of both the stiffness matrix and the mass matrix is fully exploited in the eigenvalue routine.

Numerical comparisons and examples end the paper, in which lower–upper bounds to the true results are given for a channel beam with various boundary conditions. The lower bounds were obtained by using the proposed method, whereas the upper bounds were obtained by applying for the first time—to the authors' knowledge—the Rayleigh–Schmidt method to a coupled bending torsion problem. Numerical investigations are also reported, to show the influence of the support flexibilities on the vibration frequencies.

1. INTRODUCTION

It is known from the classical beam theory that bending vibrations and torsional vibrations decouple if and only if the shear centre of the cross-section coincides with the centroid. Otherwise, coupled bending–torsional vibrations arise, with the ensuing increase of the computational cost. Moreover, a beam with thin-walled open section is also subjected to a noticeable warping, so that inclusion of the non-uniform torsion effects becomes unavoidable. Finally, quite frequently thin-walled beams have non-constant cross-sections—as, for example, in the airplane wings—and a careful design cannot neglect the flexibilities of the constraints.

The structural problem becomes rather complex, analytical solutions are not available, and finite element methods lead to large eigenvalue problems. On the contrary, in this paper a discretization method is proposed, which is enough general to allow the study of non-symmetrical non-uniform thin-walled beams with elastically flexible supports, but at the same time it is simply enough to permit extensive numerical investigations. It is also perhaps worth noting that the proposed approach furnishes lower bounds to the true results, whereas the finite element method gives upper bounds.

Let us consider a general beam with a non-symmetric cross-section, and let us fix a right-handed Cartesian coordinate system (x, y, z), as shown in Fig. 1. The origin of the system will be the centroid of the section, so that the coordinates of the shear centre are $\{x_0(z), y_0(z)\}$. The cross-section is assumed to vary arbitrarily along the z-axis, in such a way that the moments of inertia I_x, I_y along the x-axis and y-axis, the torsional rigidity C_t and the warping rigidity C_z are arbitrary—even discontinuous—functions of the z coordinate.

The beam will be discretized according to the cell method, and consequently it will be divided into a number t of rigid bars, connected together by means of $(t+1)$ elastic cells. Each cell can move along the x-axis and the y-axis, and can rotate around the z-axis. It follows that the deformed shape of the beam will be defined by the $(t+1)$ values u_i of the cell displacements along the x-axis, by the $(t+1)$ values v_i of the cell displacements along the y-axis, and by the $(t+1)$ values ϑ_i of the cell torsional rotations around the z-axis.

The structure has $3(t+1)$ Lagrangian coordinates, which can be ordered into the array $c^T = \{u_1, u_2, \ldots, u_{t+1}, v_1, \ldots, v_{t+1}, \vartheta_1, \ldots, \vartheta_{t+1}) \equiv \{c_1^T, c_2^T, c_3^T\}$.

Each cell is defined by the following four-dimensional diagonal local stiffness matrix:

$$\mathbf{k}_e = \begin{bmatrix} \frac{EI_{xi}}{l_i} & 0 & 0 & 0 \\ 0 & \frac{EI_{yi}}{l_i} & 0 & 0 \\ 0 & 0 & \frac{GC_{ti}}{l_i} & 0 \\ 0 & 0 & 0 & \frac{C_{2i}}{l_i} \end{bmatrix},$$

† On leave from Tianjin University, P.R. China.

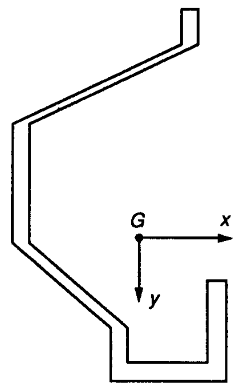

Fig. 1. General non-symmetrical cross-section.

in which E is Young's modulus, G is the shear modulus, and l_i is the length of the ith rigid bar. It is also:

$$I_{xi} = \int_{z_{i-1}}^{z_i} I_x(z)\,\mathrm{d}z, \quad i = 2, \ldots, t \tag{1}$$

$$I_{x1} = \int_0^{l_1/2} I_x(z)\,\mathrm{d}z, \tag{2}$$

$$I_{x,t+1} = \int_{l-l_t/2}^{l} I_x(z)\,\mathrm{d}z, \tag{3}$$

where $z_i = \Sigma_{j=1}^{i-1} l_j + l_i/2$. Similar formulae hold for I_y, C_t and C_2.

The distributed mass $m(z)$ can be expressed as $\rho A(z)$, where ρ is the mass density and $A(z)$ is the cross-sectional area. It can be lumped at the cells abscissae, and the following concentrated masses are obtained:

$$m_i = \rho \int_{z_{i-1}}^{z_i} A(z)\,\mathrm{d}z, \quad i = 2, \ldots, t \tag{4}$$

$$m_1 = \rho \int_c^{l_1/2} A(z)\,\mathrm{d}z, \tag{5}$$

$$m_{t+1} = \rho \int_{l-l_t/2}^{l} A(z)\,\mathrm{d}z. \tag{6}$$

Both the kinetic energy and the strain energy become quadratic functions of the Lagrange coordinates, and must be expressed in terms of the above-calculated local quantities.

2. THE STRAIN ENERGY

The strain energy of the structure can be expressed as the sum of the following quantities:

$$L = L_{fx} + L_{fy} + L_{T1} + L_{T2} + L_s$$

where L_{fx} and L_{fy} are the bending strain energies, L_{T1} is the torsional strain energy due to the De St Venant torsion, L_{T2} is the torsional strain energy due to the non-linear 'Vlasov' torsion, and L_s is the strain energy of the flexible external constraints.

2.1. *Bending strain energy*

The bending strain energy of the structure is given by:

$$L_f = L_{fx} + L_{fy} = \frac{1}{2}\sum_{i=1}^{t+1} (k_{xi}\psi_{xi}^2 + k_{yi}\psi_{yi}^2) \tag{7}$$

where $k_{xi} = EI_{xi}/l_i$, $k_{yi} = EI_{yi}/l_i$, and ψ_{xi}, ψ_{yi} are the relative rotations between the two faces of the ith cell in the x and y directions, respectively.

In order to express the relative rotations as functions of the Lagrangian coordinates, it is possible to deduce from Fig. 2 the following relationships:

$$\psi_{xi} = \frac{v_{i+1} - v_i}{l_i} - \frac{v_i - v_{i-1}}{l_{i-i}} \tag{8}$$

and:

$$\psi_{yi} = \frac{u_{i+1} - u_i}{l_i} - \frac{u_i - u_{i-1}}{l_{i-1}}. \tag{9}$$

Let us assume, for the sake of simplicity, that the length of the rigid bars is constant, and equal to l'. In this case eqns (8) and (9) become:

$$\psi_{xi} = \frac{v_{i+1} - 2v_i + v_{i-1}}{l'} \tag{10}$$

and:

$$\psi_{yi} = \frac{u_{i+1} - 2u_i + u_{i-1}}{l'}. \tag{11}$$

If these expressions are inserted into eqn (7), then it is possible, after some algebra, to arrive to the final expression:

$$L_f = \tfrac{1}{2}c_1^T \mathbf{K}_{fx} c_1 + \tfrac{1}{2}c_2^T \mathbf{K}_{fy} c_2 \tag{12}$$

where the elements of the bending stiffness matrices $\mathbf{K}_{fx}$ and $\mathbf{K}_{fy}$ can be explicitly written down in terms of

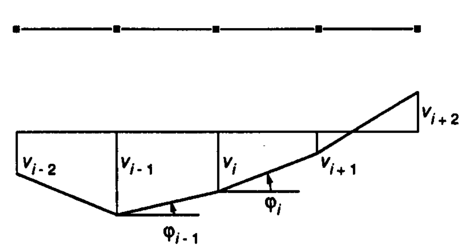

Fig. 2. Displacements and rotations of the rigid bars.

the local stiffness coefficients k_{xi} and k_{yi} [1]. It is also possible to prove that the matrices $\mathbf{K}_{fx}$ and $\mathbf{K}_{fy}$ have half-bandwidth equal to 3.

2.2. *De St Venant torsional strain energy*

The torsional strain energy of the structure, due to the uniform torsion, is given by:

$$L_{T1} = \frac{1}{2}\sum_{i=1}^{t+1} k_{\psi i}\chi_i^2 \tag{12}$$

where χ_i is the relative torsional rotation between the two faces of the ith cell, and $k_{\psi i} = C_{ti}/l_i$. It is:

$$\chi_1 = \vartheta_1$$
$$\chi_i = \vartheta_i - \vartheta_{i-1} \quad i = 2, \ldots, t \tag{13}$$
$$\chi_{t+1} = -\vartheta_t$$

so that L_{T1} can be expressed as:

$$L_{T1} = \tfrac{1}{2}c_3^T\mathbf{K}_{T1}c_3. \tag{14}$$

$\mathbf{K}_{T1}$ is a typical three-diagonal matrix, whose elements are given by:

$$K_{T1}(i,i) = k_{\psi i} + k_{\psi,i+1} \quad i = 2, \ldots, t$$
$$K_{T1}(i,i+1) = k_{T1}(i+1,i) = -k_{\psi,i+1},$$
$$K_{T1}(1,1) = k_{\psi 1},$$
$$K_{T1}(t+1,t+1) = k_{\psi,t+1}.$$

2.3. *Vlasov torsional strain energy*

The torsional strain energy due to the non-uniform torsion is given by:

$$L_{T2} = \frac{1}{2}\sum_{i=1}^{t+1} k_{2i}\Delta\chi_i^2 = \tfrac{1}{2}c_3^T\mathbf{K}_{T2}c_3 \tag{15}$$

where:

$$\Delta\chi_1 = \chi_1 = \vartheta_1$$
$$\Delta\chi_i = \chi_i - \chi_{i-1} = \vartheta_i - 2\vartheta_{i-1} + \vartheta_{i-2}. \tag{16}$$

The matrix $\mathbf{K}_{T2}$ can be calculated by inserting eqn (16) into eqn (15), and the explicit formulae are given in [1].

Finally, the global stiffness matrix will be given by:

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{fx} & 0 & 0 \\ 0 & \mathbf{K}_{fy} & 0 \\ 0 & 0 & \mathbf{K}_{T1}+\mathbf{K}_{T2} \end{bmatrix}$$

and it is a symmetric $(3t+3, 3t+3)$ square matrix with half-bandwidth equal to 3.

2.4. *The boundary conditions*

According to the general theory of the cell discretization method, the beam is supposed to be constrained by elastically flexible springs, whose stiffness can become very large, if classical perfect constraints must be simulated. This hypothesis does not seem to imply any loss of generality, because large stiffness coefficients never lead to numerical instability, and because every real structural constraint exhibits a (possibly small) flexibility.

In general, at each point of the beam it is possible to associate three elastic external 'axial' springs, with stiffness k_{vx}, k_{vy}, k_ϑ and three elastic 'rotational' springs, with stiffness $k_{\phi x}$, $k_{\phi y}$, $k_{\Delta\vartheta}$. The following non-dimensional coefficients can be conveniently defined:

$$\chi_{vx} = k_{vx}\frac{l^3}{EI_x};\quad \chi_{vy} = k_{vy}\frac{l^3}{EI_y};\quad \chi_\vartheta = k_\vartheta\frac{l}{GI_p}$$
$$\chi_{\phi x} = k_{\phi x}\frac{l}{EI_x};\quad \chi_{\phi y} = k_{\phi y}\frac{l}{EI_y};\quad \chi_{\Delta\vartheta} = k_{\Delta\vartheta}\frac{l}{GI_p}; \tag{17}$$

and a number of 'perfect' constraints can be simulated by assigning ad hoc values to these coefficients. For example, in Table 1 some torsional boundary conditions are reported, where ∞ means a large numerical value.

3. THE KINETIC ENERGY

The kinetic energy of the beam can be written as:

$$T = \frac{1}{2}\sum_{i=1}^{t+1} m_i[(\dot{y}_i + y_{oi}\dot{\vartheta}_i)^2 + (\dot{x}_i + x_{oi}\dot{\vartheta}_i)^2] + \frac{1}{2}\sum_{i=1}^{t+1} J_i\dot{\vartheta}_i^2 = \tfrac{1}{2}\dot{c}^T M\dot{c} \tag{18}$$

Table 1. Some perfect constraints

	k_{vx}	k_{vy}	$k_{\phi x}$	$k_{\phi y}$	k_ϑ	$k_{\Delta\vartheta}$
Supported along x and y	∞	∞	0	0	0	0
Supported	∞	∞	0	0	∞	0
Flexurally clamped, torsionally free	∞	∞	∞	∞	0	0
Flexurally and torsionally clamped	∞	∞	∞	∞	∞	∞
Flexural sliding	0	0	∞	∞	0	0
Flexural and torsional sliding	0	0	∞	∞	0	∞
Flexurally free, torsionally clamped	0	0	0	0	∞	∞
Flexurally free, torsionally sliding	0	0	0	0	0	∞
Supported along y-axis	0	∞	0	0	0	0

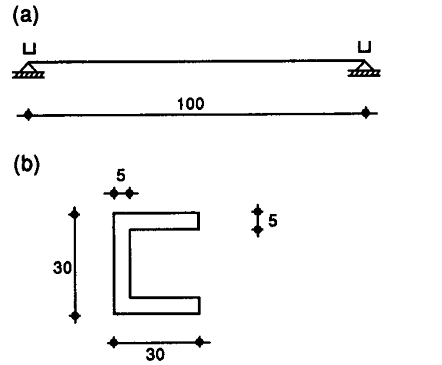

Fig. 3. Simply supported beam with channel cross-section.

where J_i is the rotatory inertia of the ith lumped mass, and (x_{oi}, y_{oi}) are the coordinates of the shear centre at the ith cell abscissa.

The Lagrangian mass matrix M will be a banded $(3t+1, 3t+1)$ square matrix, whose upper triangle is given by:

$$M(i,i) = M(t+1+i, t+1+i) = m_i \quad i = 1, 2, \ldots, t+1$$

$$M(2t+2+i, 2t+2+i) = J_i + x_{oi}^2 m_i + y_{oi}^2 m_i \quad i = 1, 2, \ldots, t+1$$

$$M(i, 2t+2+i) = m_i x_{oi} \quad i = 1, 2, \ldots, t+1$$

$$M(t+1+i, 2t+2+i) = m_i y_{oi} \quad i = 1, 2, \ldots, t+1 \tag{19}$$

4. THE EIGENVALUE PROBLEM

The Lagrangian equations lead immediately to the well-known equations of motion of an undamped unforced n-degree-of-freedom system:

$$M\ddot{c} + Kc = 0 \tag{20}$$

along with the corresponding generalized eigenvalue problem:

$$(-\omega^2 M + K)u = 0. \tag{21}$$

Both the matrices have been shown to have a peculiar highly-banded form, which greatly reduces the storage space and the running time of any convenient eigenvalue routine. All the numerical examples have been solved by using the simultaneous iteration method, as illustrated by Jennings and Corr [2], in which the matrices are stored in variable bandwidth form.

5. NUMERICAL COMPARISONS

As a first example, let us consider the single span uniform beam in Fig. 3a, whose cross-section is given in Fig. 3b. If the beam is simply supported at the ends, then the first vibration frequency can be written as [3]:

$$\omega^2 = \frac{(\omega_t^2 + \omega_b^2) - \sqrt{(\omega_t^2 - \omega_b^2)^2 + 4\lambda\omega_b^2\omega_t^2}}{2(1 - \lambda x_0)}, \tag{22}$$

where

$$\omega_b^2 = \frac{EI_y \pi^4 g}{l^4 A \gamma} \tag{23}$$

$$\omega_t^2 = \frac{(C_t \pi^2 l^2 + C_2 \pi^4) g}{l^4 \gamma (I_p + A x_0^2)} \tag{24}$$

$$\lambda = \frac{A x_0}{I_p + A x_0^2}. \tag{25}$$

In Table 2 the first vibration frequency is reported, as obtained with different discretization meshes. It is perhaps worth noting that the proposed approach leads to a lower bound to the exact values, whereas almost all the other approximate methods furnish upper bounds.

If the same beam is supposed to be clamped at both ends, then approximate solutions must be sought. For example, an upper bound can be readily obtained by using the well-known Rayleigh–Ritz method, with trial functions:

$$v(\zeta) = C\zeta^2(1-\zeta)^2 \tag{26}$$

$$\vartheta(\zeta) = D\zeta^2(1-\zeta)^2, \tag{27}$$

where ζ is the non-dimensional abscissa.

The horizontal displacement $u(\zeta)$ does not play a role, because the shear centre is on the symmetry axis of the section.

In Table 3 the first vibration frequency is reported, as obtained from the Rayleigh–Ritz method and as calculated with the authors' approach.

Finally, let us suppose that the same beam is clamped at its left end, while the right end is free (cantilever beam).

Table 2. First free vibration frequency for the simply supported beam in Fig. 3. ω_{30}^2 is the frequency given by the proposed method, by dividing the beam into 30 rigid bars, ω_T^2 is the exact Timoshenko value, ω_b^2 is the pure bending value

ω_T^2	ω_{20}^2	ω_{30}^2	ω_{40}^2	ω_{60}^2	ω_{80}^2	ω_{100}^2
696625	693961 0.380	695440 0.1701	695958 0.0957	696329 0.042	696458 0.024	696518 0.0154

Table 3. First free vibration frequency for the clamped–clamped beam

Rayleigh–Ritz 1st approx.	ω_{10}^2	ω_{20}^2	ω_{30}^2	ω_{60}^2	ω_{80}^2
3215799	2953980	3133454	3168827	3190395	3193562

In order to obtain an upper bound, two different techniques were adopted. In the first one, a classical Rayleigh–Ritz approach was used, with trial functions:

$$v(\zeta) = C_1\zeta^2 + C_2\zeta^3 + C_3\zeta^4 + \cdots \tag{28}$$

$$\vartheta(\zeta) = D_1\zeta^2 + D_2\zeta^3 + D_3\zeta^4 + \cdots. \tag{29}$$

Three approximate values were obtained, by truncating the series after the first term, the second term and the third term, respectively.

The second approach is somewhat less known, even if it is—in the authors' opinion—much more powerful. Basically, it is a modified version of the Rayleigh–Ritz method, which is referred to in the literature as Rayleigh–Schmidt method [4, 5]. The following trial functions are employed:

$$v(\zeta) = C_1\zeta^2 + C_2\zeta^n \tag{30}$$

$$\vartheta(\zeta) = D_1\zeta^2 + D_2\zeta^n \tag{31}$$

and the fundamental frequency $\omega^2 = \omega^2(n)$ is calculated. Then, the exponent value n is obtained by minimizing the frequency with respect to it:

$$\frac{d\omega^2}{dn} = 0. \tag{32}$$

In Table 4 the first free vibration frequency is given, as obtained from the Rayleigh–Ritz method, from the Rayleigh–Schmidt method, and from the proposed numerical method. The optimum n value is also reported in brackets.

From this table it is possible to deduce that the Rayleigh–Schmidt approach is quite effective, and that the proposed cell procedure gives very useful lower bounds.

Table 4. First free vibration frequency for the cantilever beam

Rayleigh–Ritz 1st approx.	184867
Rayleigh–Ritz 2nd approx.	119494
Rayleigh–Ritz 3rd approx.	116902
Rayleigh–Schmidt	117691 ($n = 2.53$)
ω_{10}^2	114904
ω_{20}^2	116356
ω_{30}^2	116630
ω_{60}^2	116748
ω_{80}^2	116818

6. NUMERICAL EXAMPLES

The influence of the torsional effects on the free vibration frequencies becomes less and less pronounced for increasing values of the ratio (span of the beam/height of the cross-section). In Table 5 the first free frequency of a simply-supported beam with channel section is given, for various values of the ratio span/height, together with the pure bending values:

$$\omega^2 = \frac{\pi^4}{l^4}\left(\frac{EI}{\rho A}\right). \tag{33}$$

All the numerical results have been obtained by dividing the beam into 30 rigid bars.

In Table 6 the same results are given for the cantilever beam and for the clamped–clamped beam. From these tables it is simple to observe that the influence of the torsional effects remains noticeable even for long beams.

The influence of the constraints flexibilities is investigated by using the structural scheme in Fig. 1. The rotational stiffness coefficients are allowed to vary from 0 to a very high value, so that various limiting cases can be recovered:

simply supported beam: $\omega^2 = 695440$;

clamped–clamped beam: $\omega^2 = 3164377$;

simply supported in bending, clamped in torsion: $\omega^2 = 1831893$;

clamped in bending, simply supported in torsion: $\omega^2 = 809148$.

It is quite clear, from this table, that the constraint flexibility plays a fundamental role in the frequency of thin-walled beams.

7. CONCLUSIONS

In this paper a recently developed discretization method has been shown to give useful lower bounds to the free bending-torsional vibration frequencies

Table 5. The influence of the ratio l/h on the first frequency for a simply supported beam

l/h	ω_{30}^2	ω_T^2	ω_b^2
0.6	4883710	4892394	23370478
1	695440	696625	3028814
3	15992	15915	37392
6	1655.18	1657.79	2336
9	388.92	389.58	461.6
15	56.000	56.100	59.828
60	0.2323	0.23274	0.2337

Table 6. The influence of the ratio l/h for a clamped–clamped beam and for a cantilever beam

	Clamped–clamped		Cantilever	
l/h	ω_{30}^2	ω_b^2	ω_{30}^2	ω_b^2
0.6	23833851	1.2009×10^8	709752	2965330
1	3168827	15563882	116630	384307
3	50544.7	192146.7	3056.66	4744.53
6	4971.093	12009.2	254.23	296.53
9	1341.75	2372.18	54.2824	58.574
15	234.386	307.43	7.36435	7.59124
60	1.16645	1.2009	0.029547	0.029653

Table 7. The influence of the support flexibilities on the first free vibration frequency

k_ϕ / $k_{\Delta\varphi}$	0	1	10	100	1000
0	695440	1714355	1819114	1830720	1831893
1	730888	1988043	2134402	2150843	2152507
10	788805	2562942	2821547	2851474	2854513
100	806942	2780370	3089301	3125427	3129101
1000	809148	2808065	3123666	3160619	3164377

of non-symmetrical thin-walled beams. A general theory is sketched, which takes into account the contribution of both the De St Venant torsion and the non-uniform torsion; the presence of elastically flexible constraints and concentrated masses can also be easily dealt with. A channel beam has been used for numerical comparisons, in which the method is shown to converge quickly to the true results. Approximate, but accurate, upper bounds have been obtained by using the Rayleigh–Ritz method and the Rayleigh–Schmidt method. Finally, the influence of the constraint flexibilities and of the ratio span/height have been numerically investigated.

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