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XXXIII. *On Reciprocal Diagrams in Space, and their relation to Airy's Function of Stress.*

Let F be any function of the co-ordinates x, y, z of a point in space, and let ξ, η, ζ be another system of co-ordinates which we may suppose referred to axes parallel to the axes of x, y, z, but at such a distance (in thought) that figures referred to x, y, z do not interfere with the figures referred to ξ, η, ζ.

We shall call the figure or figures referred to x, y, z the First Diagram, and those referred to ξ, η, ζ the Second Diagram.

Let the connection between the two diagrams be expressed thus—

$$\xi=\frac{dF}{dx}, \quad \eta=\frac{dF}{dy}, \quad \zeta=\frac{dF}{dz}.$$

When the form of F is known, ξ, η and ζ may be found for every value of x, y, z, and the form of the second diagram fixed.

To complete the second diagram, let a function ϕ of ξ, η, ζ be found from the equation

$$\phi=x\xi+y\eta+z\zeta-F;$$

then it is easily shewn that

$$x=\frac{d\phi}{d\xi}, \quad y=\frac{d\phi}{d\eta}, \quad z=\frac{d\phi}{d\zeta}.$$

Hence the first diagram is determined from the second by the same process that the second is determined from the first. They are therefore Reciprocal Diagrams both as regards their form and their functions.

But reciprocal diagrams have a mechanical significance which is capable of extensive applications, from the most elementary graphic methods for calculating the stresses of a roof to the most intricate questions about the internal molecular forces in solid bodies. I shall indicate two independent methods of representing internal stress by means of reciprocal diagrams.

First Method. Let a, b be any two contiguous points in the first diagram, and α, β the corresponding points in the second. Let an element of area be described about ab perpendicular to it. Then if the stress per unit of area on this surface is compounded of a *tension* parallel to $\alpha\beta$ and equal to $P\frac{\alpha\beta}{ab}$ and a *pressure* parallel to ab and equal to $P\left(\frac{d^2F}{dx^2}+\frac{d^2F}{dy^2}+\frac{d^2F}{dz^2}\right)$, P being a constant introduced for the sake of homogeneity, then a state of internal stress defined in this way will keep every point of the first figure in equilibrium.

The components of stress, as thus defined, will be

$$p_{xx}=P\left(\frac{d^2F}{dx^2}-\Delta^2F\right),\quad p_{yy}=P\left(\frac{d^2F}{dy^2}-\Delta^2F\right),\quad p_{zz}=P\left(\frac{d^2F}{dz^2}-\Delta^2F\right),$$

$$p_{yz}=P\frac{d^2F}{dy\,dz},\qquad p_{zx}=P\frac{d^2F}{dz\,dx},\qquad p_{xy}=P\frac{d^2F}{dx\,dy},$$

and these are easily shewn to fulfil the conditions of equilibrium.

If any number of states of stress can be represented in this way, they can be combined by adding the values of their functions (F), since the quantities are linear. This method, however, is applicable only to certain states of stress; but if we write

$$p_{xx}=\frac{d^2B}{dz^2}+\frac{d^2C}{dy^2},\quad p_{yy}=\frac{d^2C}{dx^2}+\frac{d^2A}{dz^2},\quad p_{zz}=\frac{d^2A}{dy^2}+\frac{d^2B}{dx^2},$$

$$p_{yz}=-\frac{d^2A}{dy\,dz},\qquad p_{zx}=-\frac{d^2B}{dz\,dx},\qquad p_{xy}=-\frac{d^2C}{dx\,dy},$$

we get a general method at the expense of using three functions A, B, C, and of giving up the diagram of stress.

Second Method. Let a be any element of area in the first diagram, and α the corresponding area in the second. Let a uniform normal pressure equal to P per unit of area act on the area α, and let a force equal and parallel to the resultant of this pressure act on the area a, then a state of internal stress in the first figure defined in this way will keep every point of it in equilibrium.

The components of stress as thus defined will be

$$p_{xx} = P\left(\frac{d^2F}{dy^2}\,\frac{d^2F}{dz^2} - \overline{\frac{d^2F}{dy\,dz}}\Big|^2\right), \qquad p_{yy} = P\left(\frac{d^2F}{dz^2}\,\frac{d^2F}{dx^2} - \overline{\frac{d^2F}{dz\,dx}}\Big|^2\right),$$

$$p_{zz} = P\left(\frac{d^2F}{dx^2}\,\frac{d^2F}{dy^2} - \frac{d^2F}{dx\,dy}\right),$$

$$p_{yz} = P\left(\frac{d^2F}{dz\,dx}\,\frac{d^2F}{dx\,dy} - \frac{d^2F}{dx^2}\,\frac{d^2F}{dy\,dz}\right), \qquad p_{zx} = P\left(\frac{d^2F}{dx\,dy}\,\frac{d^2F}{dy\,dz} - \frac{d^2F}{dy^2}\,\frac{d^2F}{dz\,dx}\right),$$

$$p_{xy} = P\left(\frac{d^2F}{dy\,dz}\,\frac{d^2F}{dz\,dx} - \frac{d^2F}{dz^2}\,\frac{d^2F}{dx\,dy}\right).$$

This is a more complete, though not a perfectly general, representation of a state of stress; but as the functions are not linear, it is difficult of application.

If however we confine ourselves to problems in two dimensions, either method leads to the expression of the three components of stress in terms of the function introduced by the Astronomer-Royal*.

$$p_{xx} = P\,\frac{d^2F}{dy^2}, \quad p_{xy} = -P\,\frac{d^2F}{dx\,dy}, \quad p_{yy} = P\,\frac{d^2F}{dx^2}.$$

Here
$$\xi = \frac{dF}{dx}, \quad y = \frac{dF}{dy}, \quad F + \phi = x\xi + y\eta,$$

and if ab and $\alpha\beta$ be corresponding lines, the whole stress across the line ab is perpendicular to $\alpha\beta$ and equal to $P\alpha\beta$.

* "On Strains in the Interior of Beams." *Phil. Trans.* 1863.