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Poleni's problem

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Poleni reported in 1748 on the cracked state of the masonry dome of St. Peter's, Rome. The two essential features of his analysis were that he treated masonry as a unilateral material, capable of carrying compressive forces but incapable of tension, and that his single structural consideration was that of the statics of equilibrium, with no mention of elastic deformations or of boundary conditions (compatibility). This Paper considers the structural analysis of masonry under these same restrictions, namely that the material may be idealized as rigid in compression and unilateral (thus excluding any possibility of an elastic solution), and confining structural arguments to those of equilibrium of the structure under unknown boundary conditions. These were essentially Poleni's analytical tools, and it is shown that meaningful structural statements can be made with their help. Examples are given stemming from the pathology of the masonry structure, and include the simple arch, the cross vault and the dome.

Structural criteria

It is fashionable to view structural design in terms of limit states. The jargon tends to obscure the aim of the activity—what is the problem that is being solved when an engineer undertakes the design of a structure? In fact the use of limit states does remind the designer that his structure must satisfy several of perhaps a large number of criteria. For example, a limit to permitted corrosion, or a restriction of crack width, may play leading roles in the design of a steel or a concrete frame, respectively. These two particular criteria may also play a part in the design of masonry, although it seems reasonable to suppose them to be of secondary importance, to be reviewed finally by the designer but not likely to dictate the design.

2. The three main structural criteria are those of strength, stiffness and stability. The structure must be strong enough to carry whatever loads are imposed, including its own weight, it must not deflect unduly, and it must not develop large unstable displacements, either locally or overall. If these three criteria can be satisfied, the designer can run through a check list of secondary limit states to make sure that his structure is serviceable.

Behaviour of masonry

3. An immediate, and paradoxical, difficulty arises when the ideas of strength, stiffness and stability are applied to masonry. An ancient structure—the Roman Pantheon, for example, or a Greek temple—seems intuitively to be strong enough, in the sense that the loading (self weight and wind) has not, over the centuries, caused failure to occur by fracture of the material. This matter will be discussed further, but it is a fact that mean stresses are very low in a typical masonry structure; local spalling may be seen, but this seems hardly to affect the structural integrity of the whole.

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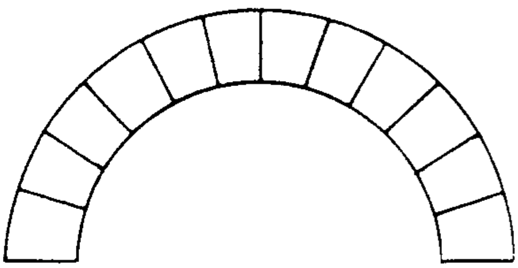
4. Similarly, the engineer is unlikely, in the first instance, to worry about unduly large working deflexions of the vault of a Gothic cathedral. Strength and stiffness do not lie in the foreground of masonry design; it is the third major criterion, that of stability, that is relevant, albeit in a curious form. The masonry arch of Fig. 1, for example, may be perfectly comfortable under the action of its own weight and at a certain intensity of the superimposed point load P. The stresses are low and the deflexions negligible, and both will remain so as the value of P is increased. However, at a certain value of P the arch can no longer contain the structural forces, and an unstable mechanism of collapse will form (the four-bar chain of Fig. 1(*b*)). How this happens is discussed later.

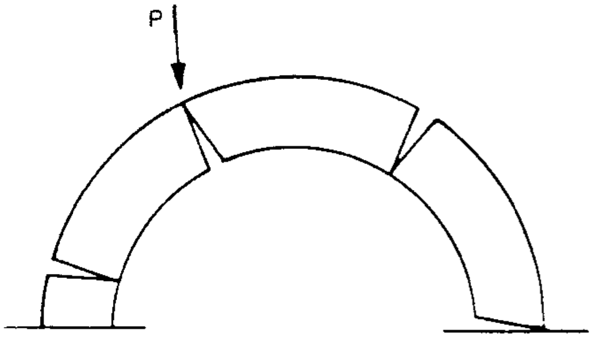
5. The semicircular arch of Fig. 1 will carry a given load P provided that the arch ring has a certain minimum thickness, and the design of the arch consists in somehow assigning a thickness for a given span. Once this design has been made and the arch constructed, then it will at once, and ever after, satisfy the criterion of strength (it will not crush), of stiffness (deflexions will be negligible) and stability (a four-bar chain will never develop). The design consists, curiously to the mind of the modern engineer, in assigning correct proportions to the arch.

6. It was precisely rules of proportion that were used by classical builders to design their structures, as is at once evident, for example, from Vitruvius. These rules were never lost:[2] they survived the Dark Ages, were built in to the secret books of the masonic lodges, and flourished in the 12th and 13th centuries in the age of High Gothic, and later, until the whole apparatus was swept away by the Renaissance.

Modern analysis

7. The end of medieval structural design was heralded by one of Galileo's *Two new sciences*.[3] In 1638 he posed the problem of the assessment of the strength of a


(a)



(b)

Fig. 1. Collapse mechanism for a masonry arch

cantilever beam: what was the value of the breaking load? This sort of question may have been asked before (a tree trunk across a ditch will break under a heavy load), but not in the context of the development of practical design rules. Galileo wished to determine the strength of the transversely loaded beam as a function of its breadth and depth, so that a formula could be devised from which the strength of any (rectangular cross-section) beam could be calculated. This is an example of what a modern engineer recognizes as the design process (for strength of his structure).

8. The new science of structural mechanics was eagerly pursued in the 18th century, when solutions to four main problems were sought—the strength of beams (Galileo's problem), the strength of columns, the thrust of arches (against abutments, say river banks) and the thrust of soil (against retaining walls). Advances were slow, being held up (among other things) by lack of a proper understanding of first-order tensors, i.e. vectors. Ideas of stress slowly emerged but, in turn, these could only develop properly after the second decade of the 19th century, when the second-order (stress) tensor had been worked out.

9. Two centuries after Galileo the problem of the breaking of a cantilever beam had been transformed into a problem of the determination of the value of the stress in that beam. Side by side with these advances through rational mechanics, experiments on commonly used building materials had established reference values of limiting stresses. It was a natural step to attempt to relate the two values, to try to arrange that the working values of the stress should have an adequate margin of safety when compared with the limiting values for the materials used. Navier[4] seems to have been the first to declare that the engineer was not in fact interested in the collapse state of his structure (that is, in answering Galileo's question), because all were agreed that it was a state to be avoided. Rather, the engineer would wish to be assured that the working state was safe, and this assurance would be given by limiting the actual stresses in the structure to a proportion of their ultimate values. It is, implied Navier, the engineer's job to calculate this working, or actual, state, and then to use his rules and formulas to design the structure. *Prima facie* this seems a sensible procedure, but doubts arise when the analytical process is examined in detail.

10. The designer wishes as a first step to find the internal forces in his structure (thrusts, bending moments and so on). The first equations written are those of statics; the internal forces must be in equilibrium with the loads imposed on the structure. If the engineer can solve these equations straight away the first step is complete (and, technically, the structure is statically determinate). Generally, however, the equilibrium equations, standing alone, are insoluble; the structure is statically indeterminate (hyperstatic). There are many possible equilibrium states, and to determine the actual state requires the use of other equations of structural analysis.

11. For example, the arch of Fig. 1(*a*), carrying only its own weight, has infinitely many equilibrium configurations. Hooke 'solved' the arch problem in 1675 with his statement 'as hangs the flexible line, so but inverted will stand the rigid arch,' although he was unable to give a mathematical description of this powerful theorem. Poleni knew of this principle, and Fig. 2 shows his sketch of a chain hanging in tension under its own weight; the same shape, inverted, gives the trace of an arch which will carry the same loads in compression. The geometry of this thrust line will depend on the actual length of the chain, and the distance apart of the supports. Thus a possible inverted chain is shown in Fig. 3(*a*) lying within

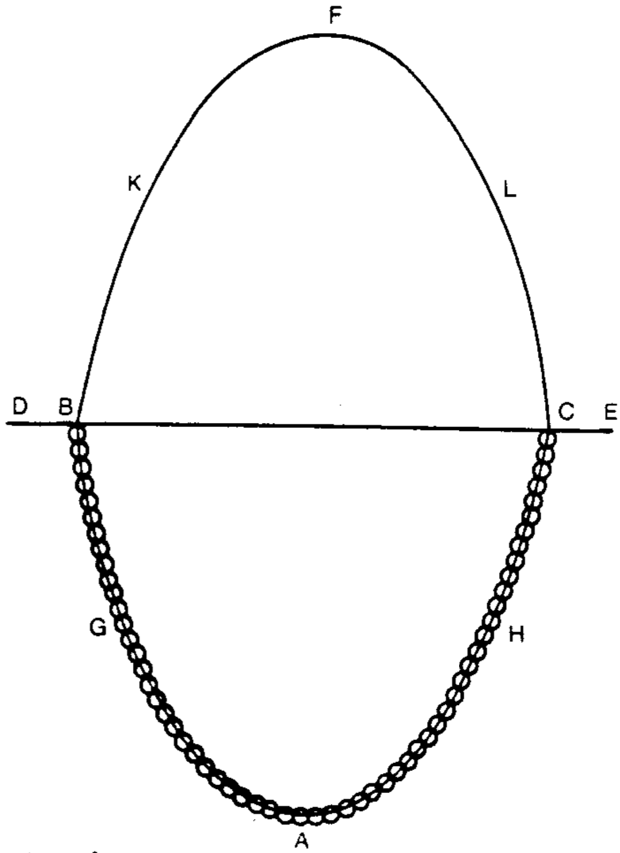


Fig. 2. Hooke's hanging chain

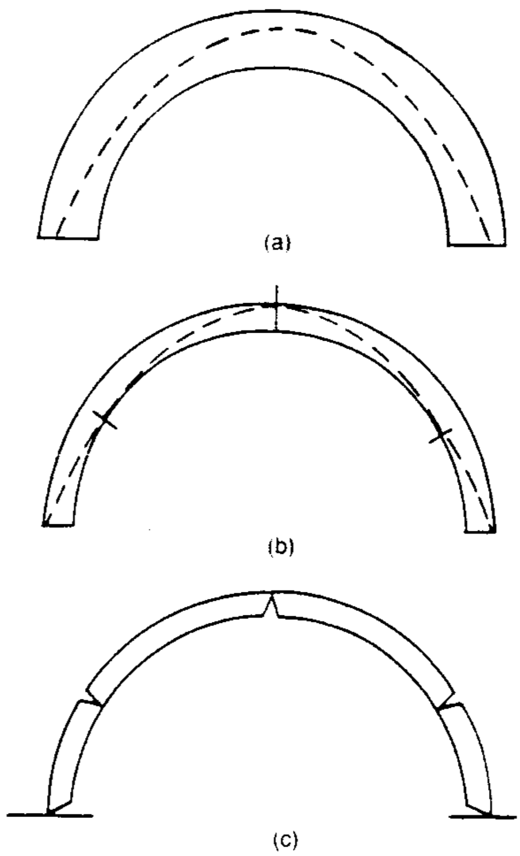


Fig. 3. Arches: (a) stable and (b), (c) arch of minimum thickness

the boundaries of the semicircular arch. For masonry, as will be seen, thrusts must lie within the external boundaries of the construction, and it is clear that many other catenaries could have been drawn between the extrados and the intrados of the arch of Fig. 3(*a*).

12. Thus the equation of equilibrium, taken by itself, does not give sufficient information to determine the actual position of the thrust line in Fig. 3(*a*). The other two equations of structural analysis must be used. The first equation represents a statement of material properties, so that internal deformations of the structure can be related to the internal forces. The second equation represents a geometrical statement, and normally arises as some internal or external constraint; in Fig. 3(*a*), for example, the arch rests firmly on rigid foundations, and the internal deformations of the arch must be such that the boundary conditions are satisfied, namely that displacements are zero at the abutments.

Elastic design

13. Navier's design philosophy involves the postulation of an elastic law of deformation and the assumption of certain boundary conditions that arise in the solution to the problem. In this (formal) way enough equations will be available for the solution of the hyperstatic problem; if the material of the arch of Fig. 3(*a*) is elastic, and if the abutments are rigid, a unique position for the thrust line equilibrating the loads (self-weight or any other specified load) can be calculated.

14. The solution of the structural equations for a hyperstatic structure is extraordinarily sensitive to very small variations in the boundary conditions. If one of the fixed abutments of the arch in Fig. 3(*a*) should suffer a small displacement, this would be accompanied by a large shift in the position of the thrust line. (Small in this connection implies a displacement containable within the thickness of the lines in Fig. 3(*a*). The eye would detect no difference between the perfect and displaced states.) Now it is certain that one of the abutments of the arch will suffer an (unpredictable) small displacement, and this then puts in question the whole rationale of elastic analysis.

15. Elastic analysis attempts to answer the question: what is the actual state of the structure? If the structure is hyperstatic, the analysis must take into account the material properties, and it must make some assumption about conditions of compatibility of deformation—for example, boundary conditions at the abutments of an arch. If the abutments were known to be absolutely rigid, or if they were known to behave in a certain predictable way, the actual state of the structure could indeed be calculated. As it is, the actual state is ephemeral; a severe gale, a slight earth tremor, a change in water table will produce an entirely different equilibrium state for the structure.

16. It was this situation that was observed by the Steel Structures Research Committee (SSRC)[5] in their series of tests on real steel-framed buildings in the early 1930s. The stresses (actually strains) measured in practice bore almost no relation to those confidently calculated by the designer. They concluded that the problem could not be avoided—practical imperfections of construction and behaviour were inevitable, and would always lead to an unpredictable working state of a structure. The structural profession was in fact forced to look for an alternative method of design.

Plastic design

17. Common sense would seem to indicate that a severe gale, easily survived

by a building, but leading to a completely different state of the structure, cannot really have weakened the structure. Common sense is in this case supported by both theory and experiment. If two seemingly identical structures, but actually with different small imperfections so that they are in very different states of initial stress, are loaded slowly to collapse, the collapse loads (that is, the strength of the structures) will be found to be the same. It was this observation which led to the development of the plastic theory of structures, applicable to any case where collapse is a ductile quasi-stable process (and therefore to steel and to reinforced-concrete frames, and to any building type using a structurally common material (not cast iron or glass, which are brittle), and which does not develop local or overall instability).

18. The plastic designer does not, of course, envisage that his structure will actually collapse. He performs a calculation based upon loads increased by a hypothetical factor, and then states that the structure acted upon by the (smaller) working loads will never collapse. This statement stems from the master theorem of plastic analysis, and is the key tool of the designer.

19. The plastic designer is not really working with collapse loads. He constructs an equilibrium state under the working values of the loads, and then proportions the members of the structure so that the internal forces associated with the equilibrium state can be carried safely. In this sense the plastic designer and the elastic designer have reached common ground. The latter believes that he has established the actual working state in equilibrium with the applied loads. The plastic designer, on the other hand, does not attempt to answer the question: what is the actual state? He establishes a particular (and, from the point of view of design, very favourable) state which is in equilibrium with the applied loads. Both designers then proceed in almost exactly the same way to detailed design of the members. The plastic designer's route is more direct: using the (idealized) stress/strain curve of Fig. 4(*a*), in which an elastic response is followed by a perfectly plastic limit, he can predict the collapse limit state more easily than the elastic designer can calculate the actual working state.

20. The elastic properties of the material were introduced, as has been seen, so that the actual state of a hyperstatic structure could be calculated. Thus elastic properties are present in the stress/strain curve of Fig. 4(*a*), although no use of them is made by the designer. A greatly simplifying step is then available: the elastic/plastic idealization of Fig. 4(*a*) can be replaced by the rigid/plastic idealization of Fig. 4(*b*). No information is lost to the plastic designer, because he makes no pretence of calculating the elastic working state, and either Fig. 4(*a*) or 4(*b*) can

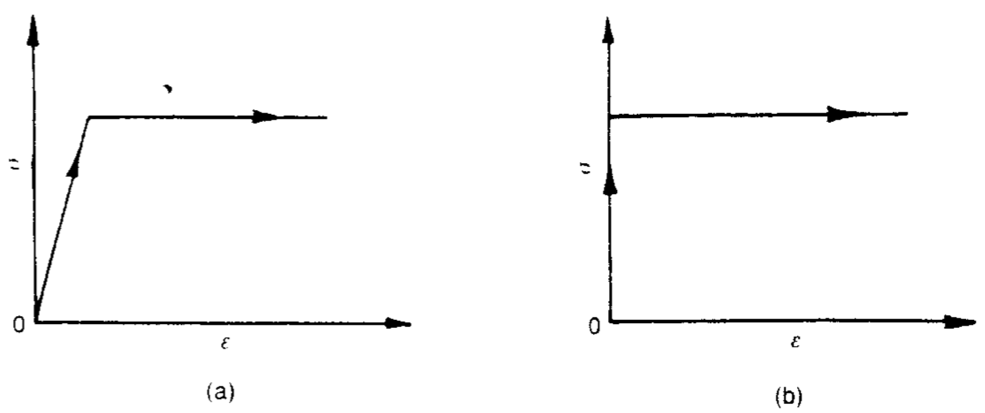


Fig. 4. Stress/strain curves: (a) elastic/plastic, (b) rigid/plastic

be used to give the same prediction of the plastic limit state. However, the curve of Fig. 4(*b*) cannot possibly give any information about the actual state of the structure.

21. What the designer can do is to construct an equilibrium solution and apply the master safe theorem of plasticity. If any equilibrium state can be found, that is, one for which a set of internal forces is in equilibrium with the external loads, and, further, for which every cross-section of the structure satisfies a strength criterion (for example, that the stresses at every section of the structure are less than the yield stress of the material), the structure is safe. The power of this safe theorem lies in the fact that the equilibrium state examined by the designer need not be the actual state; viewed anthropomorphically, the safe theorem declares that if the designer can find a way in which the structure behaves satisfactorily, the structure itself certainly can.

22. There is thus no need to attempt the calculation (say an elastic calculation) of the actual state of the structure, and this is the reason that no information is lost about the strength of the structure by idealizing the stress/strain curve of Fig. 4(*a*) to the rigid/plastic curve of Fig. 4(*b*). It may be noted in passing that the elastic design process is, by these arguments, safe. It does not calculate observable quantities, but nevertheless generates an equilibrium solution, and the designer ensures that stresses are everywhere below limiting values. By the plastic theorems, therefore, the elastic process is safe.

23. As a matter of interest, the way the plastic designer constructs his equilibrium solution is as follows. He arranges for his structure to collapse under the specified working values of the loads. In simple steel design, this will furnish a set of values for the full plastic moments of a frame. The designer then arranges that the structure, as built, is stronger by an agreed margin: that the full plastic moment at every section, for example, is 1·75 times the value required for incipient collapse under the working values of the loads. The value of 1·75 is usually referred to as the collapse load factor, but it can be seen, alternatively, as the margin of strength possessed at every section by the frame as built as against one that would just stand under the working loads. It is possible to transfer this plastic philosophy to an examination of the behaviour of masonry.

Design of masonry

24. Masonry is a unilateral material. It is a collection of bricks or stones, squared or unsquared, or of sun-dried mud (adobe), assembled dry or with weak mortar into a coherent whole. As such, the resulting structure can transmit compressive forces, but can resist only feeble tensions; any attempt to impose tensile forces will result in the assemblage pulling apart. Thus the idealized (unilateral) stress/strain curve will be as shown in Fig. 5(*a*); there is an elastic response in compression, but tensile strains can develop with zero stress.

25. The stress/strain curve of Fig. 5(*a*) can be introduced as a complicating modification to the usual process of elastic analysis, and this is precisely what Castigliano[6] did as a sophisticated illustration of his energy theorems. The three equations (static equilibrium, compatibility of deformation, and material properties) can be combined in a strain–energy formulation (strictly, complementary energy) for the solution of redundant structures. In this formulation, or indeed in the direct solution of the basic equations, the section properties of the structural members are needed. Thus, in his determination of the forces in a large-span masonry bridge (Ponte Mosca, Turin), Castigliano used the second moment of

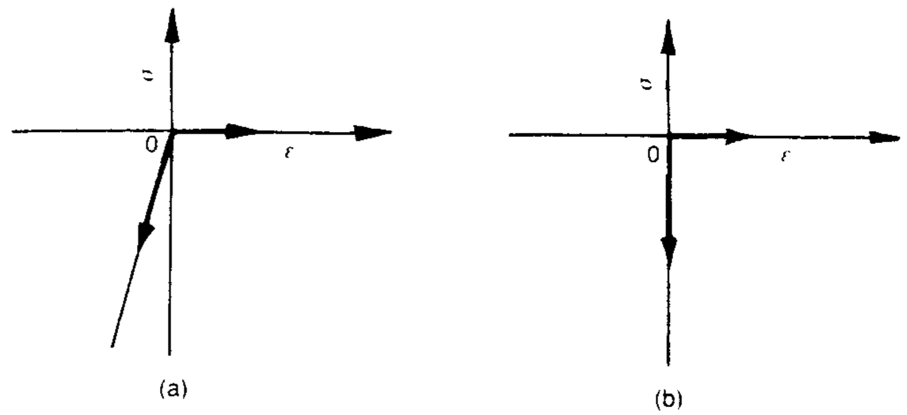


Fig. 5. Stress/strain curves for unilateral masonry: (a) elastic in compression, (b) rigid in compression

area of the rectangular cross-section of the arch ring. So long as the line of thrust in the arch ring lies within the middle third the whole of the cross-section will be in compression; if the line of thrust lies outside this central core, but still within the arch, a normal elastic material will develop tension. Unilateral masonry will crack.

26. In Castigliano's first trial at a solution, the thrust line did indeed lie outside the middle third for portions of the arch. Using this first position of the thrust line as an estimate of its final position, Castigliano calculated the extent of cracking, the corresponding reduction in depth, and hence a modified second moment of area for each cross-section of the arch. A new analysis using this modified arch led to a slightly different position of the thrust line, and the analysis was repeated. Convergence is rapid.

27. Castigliano, working with a unilateral material, was attempting to answer Navier's question: what is the actual state of the structure? In making his analysis he made the usual kind of assumption about the boundary conditions on deformation—in this case that the abutments were rigid. The analysis is therefore subject to the same objections as a conventional elastic analysis and, in particular, it will bear little relation to the observable condition of the arch if the abutments have been subject to movements, which need have been only slight.

28. No large-scale measurement of internal forces in masonry structures has been attempted, to parallel the SSRC's investigation of steel frames in the 1930s. If such tests were carried out, they would surely mirror the findings of the SSRC, and elastic (or Castigliano's unilateral elastic) calculations would be found to be poor predictors of the actual state of a structure. As for steel, a better understanding of the behaviour of masonry is obtained from a plastic viewpoint.

Plastic theorems for masonry

29. It is convenient to present the basic principles of the action of masonry in terms of a regular structure, say the arch of Fig. 1 formed of simple voussoirs. However, the conclusions apply to any, and much more complex, forms of masonry construction, say a complete Gothic cathedral.

30. As has been mentioned, mean compressive stresses in masonry are usually extremely low. In a large-span bridge or in one or two elements of a cathedral (crossing piers bearing the weight of a tower) stresses may approach one tenth of the crushing strength of the material. The main structural elements will be working

at about one hundredth of the crushing stress, and subsidiary elements at one thousandth. Following these indications, the assumption will be made that, in general, stresses are so low that there is no danger of failure of the material, which can therefore be taken to be infinitely strong in compression. On the other hand, weak mortar, or interlocking of the stones, cannot be relied on to generate tension in the construction. It will be assumed that the material is unilateral, unable to support tensile stress. Finally, it will be assumed there is a sufficient level of ambient compressive forces between portions of the masonry that friction prevents sliding of one part against another.

31. None of these assumptions is strictly true, and examples of their violation can be found occasionally in practice, and can be allowed for in the analysis of the structure. However, the assumptions (infinite compressive strength, no tensile strength, no slip) enable the plastic thorems to be translated directly from steel to masonry.

32. The master 'safe' theorem can be 'translated' with reference to Hooke's inverted chain. The equilibrium statement is represented by this line of thrust; as the forces in masonry must be compressive, the inverted chain is constrained to lie within the boundaries of the masonry. The safe theorem then states that if any one such position can be found for the line of thrust, this is an absolute proof that the structure is stable, and indeed that collapse can never occur.

33. In terms of Hooke's analogy, the geometry of the real arch must enclose one, at least, of the infinite number of inverted chains which correspond to the given loading. Once again, the actual state of the structure is of no final relevance to this assurance of stability; the accidental small displacement of an abutment cannot be seen, within the thickness of a pencil line, on the drawing board, and as the geometry is virtually unaltered, the stability is unimpaired.

34. The implication here is that cracking of masonry is not, in essence, dangerous. Small imperfections may lead to an apparently drastic response of the structure, and to visible cracks where the thrust line (to think of Castigliano's arch) has passed outside the middle third, but still lies within the masonry. Masonry, however, as a unilateral material, is supposed to crack. Provided that movements are small, they do not portend a ruinous state for the structure as a whole. They are symptoms of the natural morbidity of masonry. Some of the pathology is discussed below.

35. The question arises as to what, if any, meaningful statements can be made about the masonry structure when the tools of analysis are grossly simplified, and when some information is discarded. Specifically, of the three basic structural statements, that of the statics of equilibrium must be retained. If to this is added the idea of a unilateral material, the powerful safe theorem applies. The structure can never collapse, despite any appearance of cracks, and despite any future random small displacements of its abutments.

36. Further, the safe theorem will continue to apply if the unilateral stress/strain relation of Fig. 5(*a*) is idealized to that of Fig. 5(*b*); the material is rigid in compression, and can accept no tensile strains. Thus no elastic constants are specified, or rather the elastic modulus is zero in tension and infinite in compression, and there is therefore no possibility of obtaining any actual estimate of the state of a structure. Finally, as the condition of the connection of the structure to its environment is essentially unknowable, it will be taken as unknown; no assumptions about rigidity of abutments, for example, will be made.

37. Formally, then, a structural analysis of masonry is to be made with the

following use of the three basic statements of structural theory

(*a*) Equilibrium: the use of statics establishes that the structure is permanently stable, even though its actual state may differ from time to time

(*b*) Material behaviour: the material is unilateral, idealized as rigid in compression; therefore no elastic response is calculable

(*c*) Compatibility of deformation: as calculations of the actual state are not being made, equations of deformation are not needed. In any case the analysis will proceed on the assumption that boundary conditions are unknown.

Under these three restrictive conditions, it turns out that meaningful statements can be made about the safety of a masonry structure and, further, that an attempt can be made to interpret the defects exhibited in practice by certain structural elements.

Geometrical factor of safety

38. The thrust line sketched in Fig. 3(*a*) is only one of an infinite number of such lines. If the weight of the arch is distributed uniformly, the shape of the thrust line is that of the catenary and is of course geometrically different from that of the circular arch. Nevertheless the arch of Fig. 3(*a*) has sufficient depth to accommodate a range of thrust lines. There is a minimum thickness of semicircular arch (just over 10% of the radius) which will just contain a catenary, and this is sketched approximately to scale in Fig. 3(*b*). A thinner arch cannot be constructed without the equilibrating thrust line passing outside the masonry, which would imply tension in contradiction to the unilateral assumption of Fig. 5(*b*). The arch of Fig. 3(*b*) is on the point of collapse by the formation of the five-bar chain of Fig. 3(*c*); any slight asymmetry (of geometry or of loading) will suppress one of the abutment hinges and turn the mechanism into a regular four-bar chain.

39. The arch of Fig. 3(*b*) is containable within the arch of Fig. 3(*a*), and this leads to the idea of a geometrical factor of safety, defined by the ratio of the thicknesses of the two arches. If this factor were 3, for example, the thrust line for the arch of Fig. 3(*a*) could be contained within its middle third (which is, of course, a purely elastic concept); if the factor were 2, the thrust line could be contained within an arch of half the real depth, and so on. (Safe practical values for the geometrical factor of safety seem to be about 2.)

40. The idea of a geometrical factor in masonry construction therefore replaces the idea of a strength factor in the straightforward application of simple plastic theory to steel or concrete frames. For frame design to a factor of 2, the full plastic moment would be calculated for collapse just to occur under the working loads, and the real frame would be built with twice this strength. In masonry design to a factor of 2, the thickness would be calculated for collapse just to occur under the working loads, and the real structure built with twice this thickness.

41. There are other, numerical, statements which can be made. For example, the two positions of the thrust line in Fig. 6 are certainly the extreme positions, and each corresponds to a calculable value of the horizontal component of the abutment thrust. The two values of this thrust represent limits between which the actual value must lie at any time, so that the designer can carry out abutment design with some confidence.

42. Thus, despite the apparent extreme restrictions of the analysis, strong

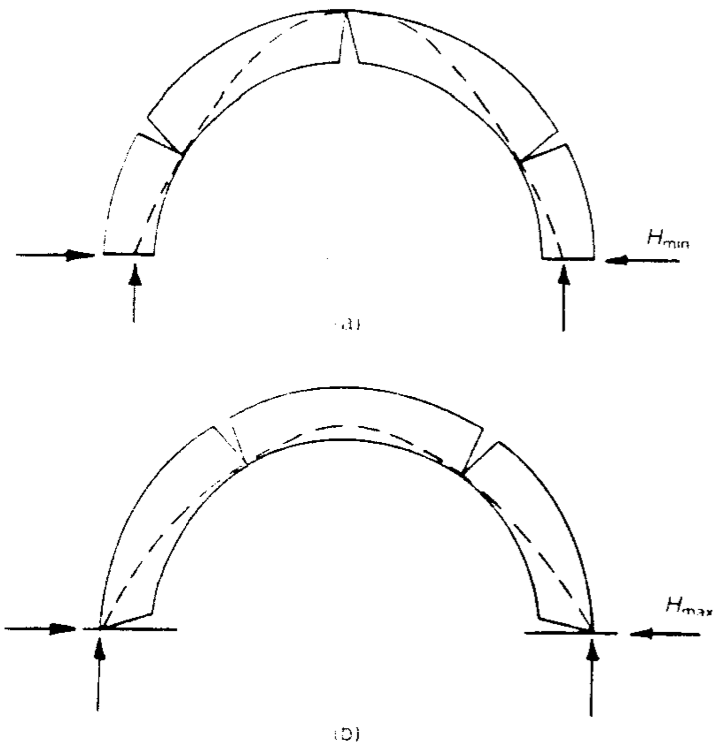


Fig. 6. Extreme positions of the thrust line

statements can be made. First, a numerical value can be calculated which represents, geometrically, the safety of a masonry construction. Second, structural quantities of design significance (for example, values of abutment thrust) can be determined to lie within calculable bounds.

Some pathological studies

43. A structure made from rigid/unilateral material (the stress/strain curve of Fig. 5(*b*)) cannot deform without cracking. It is of interest to attempt to predict the location of such cracks which will develop as a result of small movements of the environment which supports the structure. Conversely, it is of interest to make a pathological examination of a cracked structure, and to attempt estimates of the movement of the environment.

The arch

44. As a simple example, the segmental circular arch of Fig. 7(*a*) might be supposed to be built, perfectly, on centring. When the centring is removed the arch will thrust against its abutments and these, inevitably, will give way. To accommodate itself to the increased span the arch must crack, and it will do so in the pattern of Fig. 7(*a*). This then will be the expected response to a small but unknown outward movement of the abutments; the resulting three-pin arch is stable, is in a statically determinate configuration, and is thrusting with its now calculable minimum value of abutment thrust.

45. Had the arch been pointed, Fig. 7(*c*), the line of thrust would have been unable to 'reach' the crown, Fig. 7(*d*), and the single central hinge would, theoretically, be replaced by two displaced hinges. In practice, slight asymmetry will

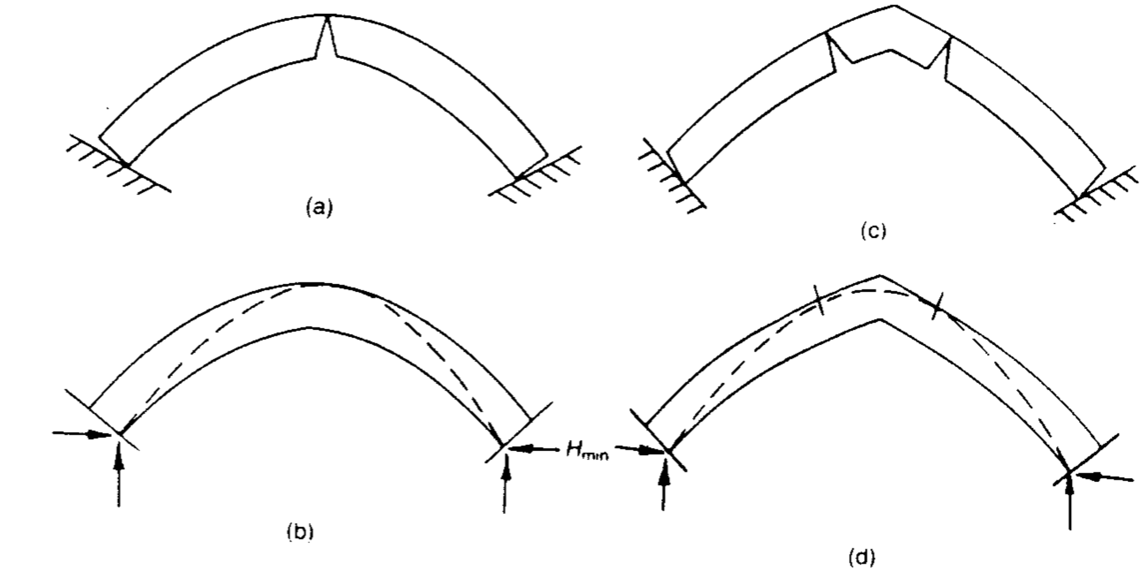


Fig. 7. Cracking of arches due to increase in span

ensure that only one of the two hinges near the crown will form. As all arches thrust, and all river banks give way, it is a common sight when passing under a masonry bridge to see a single crack running along the barrel of the arch at or near the crown.

The cross vault

46. A simple extension of these basic ideas for the arch leads to an understanding of the mechanics of the vault.[7] As a first step towards this understanding, Fig. 8(*a*) shows the cross-section of a uniform cylindrical barrel vault, drawn roughly to scale (say a vault thickness of 0·3 m with a span of 15 m). The vault is supposed to be maintained by the external buttresses, with or without flyers spanning side aisles. As drawn, the vault is in fact too thin to carry its own weight. The minimum thickness of 10% of the radius, with no margin of safety (Fig. 3(*b*)) implies that the vault must be thicker than 0·75 m. Fill is therefore shown in Fig. 8(*a*) backing the haunches of the barrel, allowing a pathway for the vault forces in their 'escape' from the vault proper.

47. The external buttressing system will, almost inevitably, give way slightly under the imposed thrusts from the vault, and movements will occur during the first few years of the life of the structure. Movement of the buttresses will virtually cease when the soil under the foundations has consolidated under the bearing pressures. The crack pattern of Fig. 8(*b*) has already been noted (cf. Fig. 7(*a*)). Had the barrel vault had a pointed cross-section, the crack near the crown of the vault

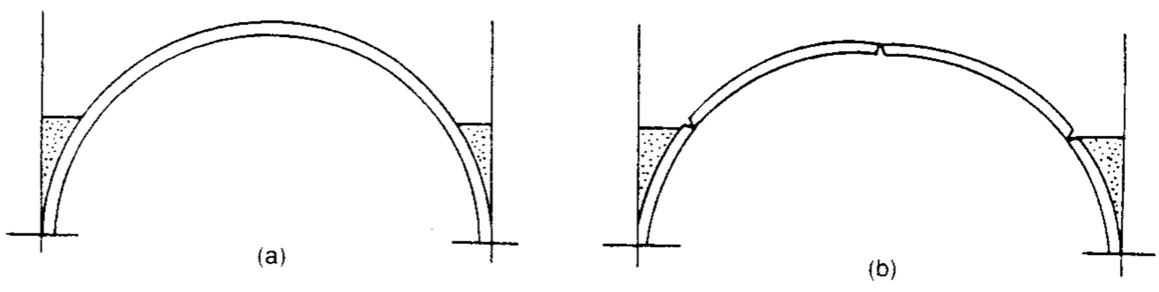


Fig. 8. The cylindrical barrel vault: (a) undeformed, (b) cracked

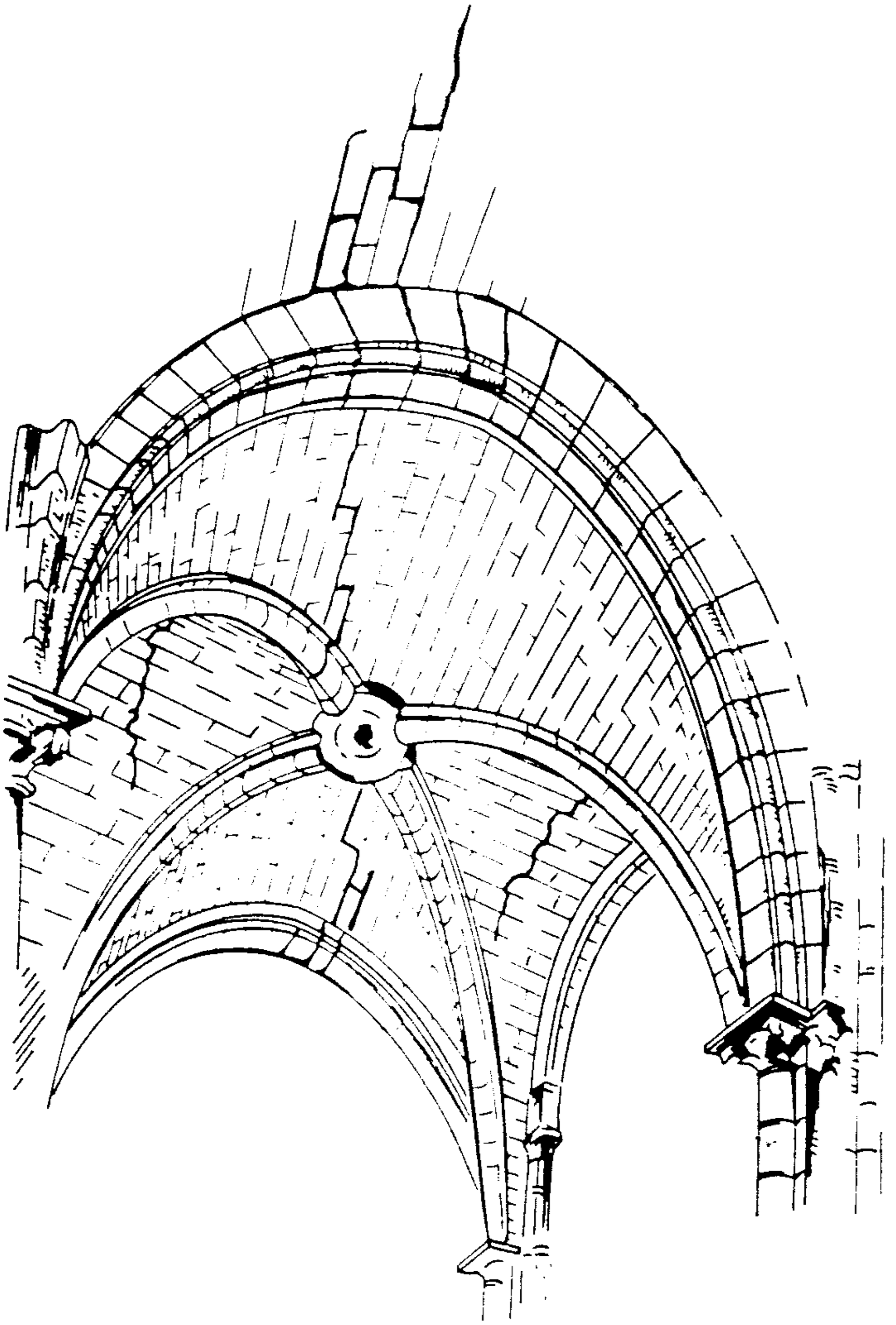


Fig. 9. A cracked quadripartite vault (from Abraham)

would have been off centre. (cf. Fig. 7(*c*)). Fig. 9, from Pol Abraham,[8] shows a sketch of typical cracks of this sort near the crown of a slightly pointed quadripartite vault. This is a first kind of chronic defect exhibited in practice by masonry vaults, but other cracks are also visible in Fig. 9.

48. The vault of Fig. 8 is essentially two-dimensional, in the sense that the cross-section was supposed to be the same down the length of the church. Fig. 10 shows, schematically, a single square bay of a quadripartite vault formed by the intersection of cylindrical barrels. In Fig. 10(*a*) an elevation of the vault is shown,

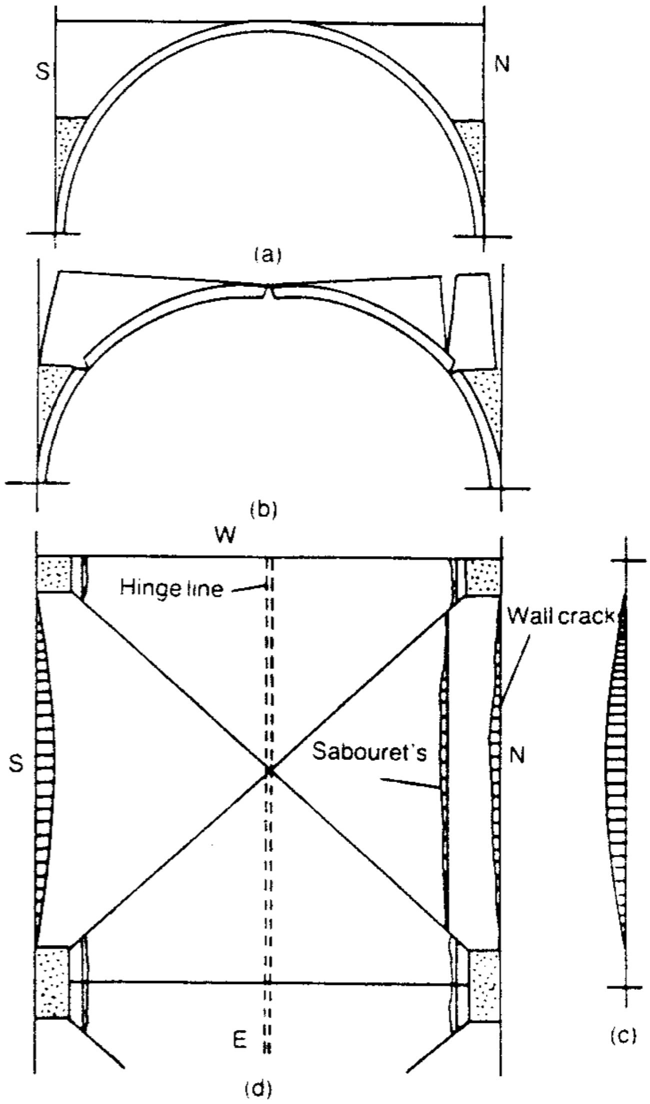


Fig. 10. Schematic quadripartite vault

looking west down the length of the church; the fill, which serves the same function as before, is placed in the vaulting conoids (cf. the plan of Fig. 10(*d*). The vault is of course supposed to extend for several bays, as indicated in the plan). If, now, the buttressing system of the vault gives way, that portion which runs east-west will crack as before, Fig. 10(*b*), and the single hinge line at or near the crown, the first kind of chronic defect, will be visible from within the church. The change in geometry is accommodated by rotation of the three hinges, with a consequent drop of the crown of the vault.

49. There is, however, a severe geometrical constraint imposed on the intersecting vault which runs north/south. The horizontal soffit of this vault was built to the original dimension of the span, but the span has now increased. At the plane of the buttressing system the mismatch in dimensions is zero, but it increases to a maximum at the crown, as indicated in Fig. 10(*c*). The general effect of this mismatch is evident in the elevation of Fig. 10(*b*); from the left-hand (south) side of the plan in Fig. 10(*d*) it will be seen that the whole of the geometrical incompatibility could be taken up by a gap opening between the vault and the wall.

50. However, this would place the masonry of the north-south vault adjacent to the wall in a severe state of strain. A crack pattern which allows the vault to deform in more or less strain-free monolithic pieces is sketched in the right-hand (north) side of the plan in Fig. 10(*d*). A secondary crack (Sabouret's crack in Pol Abraham's taxonomy, after the French architect[9] who drew attention to its significance) has opened parallel to the wall crack (cf. Fig. 9). A Sabouret crack and a wall crack will effectively isolate a portion of the north-south vault, and this portion will then be free to act as a simple arch running east-west, and spanning roughly between adjacent vaulting conoids (although the width in practice of this isolated arch will depend also on other factors, including the shape of the main cross-section of the vault).

51. Thus cracks near the crown are traces of hinge lines in a portion of the vault through which compressive forces are being carried, the forces in fact acting approximately perpendicular to the hinge lines. By contrast the wall cracks and Sabouret's cracks represent complete separation of the masonry, through which a hand may often be passed. No forces can be transmitted across such fissures, and the compressive forces run parallel to the cracks.

52. The behaviour modelled in Fig. 10 was based on an idealized quadripartite cylindrical vault having square bays. The general pathology of all vaults is of the same nature, and can be observed in groin and rib vaults with rectangular bays, and indeed in fan vaults. The isolation of several compartments of the vault by crack patterns leads to an immediate qualitative understanding of the way the forces are being carried, and this can then be supplemented by more or less precise calculations of their magnitudes.

The dome

53. Poleni's report of 1748 was commissioned to account for the cracked state of the dome of St. Peter's, Rome—this is Michelangelo's dome of about 1546, but built after his death by Fontana and della Porta in about 1570. The dome was exhibiting a large number of meridional cracks, running up from the base to near the crown, and the questions to be answered, then as now, were: what is the meaning of these cracks? Are they dangerous?

54. Poleni gives a comprehensive review of the mid-18th century state of knowledge of masonry construction. As has been noted, Fig. 2, he knew of Hooke's hanging chain, and he has an astonishing preperception of the safe theorem by his explicit statement that stability of a structure is assured if it can be established that the line of thrust lies completely within the masonry.

55. Poleni noted that the cracks of St. Peter's had already divided the dome into portions approximately half spherical lunes (orange slices); for the purpose of his analysis, he sliced the dome hypothetically into 50 such lunes. He then considered the stability of, effectively, a two-dimensional arch composed of two lunes leaning against each other at the crown; half such an arch is shown schematically in Fig. 11. From a drawing of the cross-section of the dome Poleni calculated the weight of the lune, dividing it into 16 segments for this purpose, and making due allowance for the weight of the lantern. He then threaded a flexible string with 32 weights, each weight in proportion to a segment of the two lunes, and so determined experimentally the shape of the hanging chain. This thrust line, superimposed on the original drawing, did indeed lie within the boundaries of the masonry of the dome. Poleni concluded, correctly, that the hypothetically sliced dome was stable and that, *a fortiori*, the original dome, cracked or uncracked, was also stable.

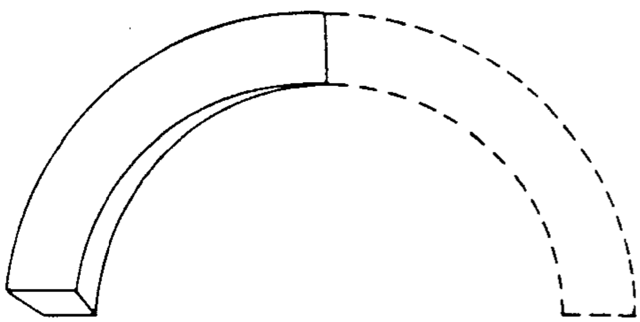


Fig. 11. Two-dimensional 'orange-slice' arch cut from a dome

56. Poleni had in fact established a solution for the dome for which the hoop stress is zero. By considering the notionally separate lunes he had found a way in which the whole dome could satisfy the conditions of equilibrium (although, as will be seen, the solution must be modified in the neighbourhood of the crown if the dome has an eye). The same technique can be used to obtain a general understanding of the way in which a masonry dome behaves, and, just as the circular arch gives some insight into the behaviour of any arch, so the uniform spherical dome repays study.

57. Figure 3(*b*), for example, shows the arch of constant width which has a minimum thickness. The orange-slice arch composed of two lunes, Fig. 11, may be treated in the same quasi-two-dimensional way, and the corresponding minimum thickness for the dome can be calculated. Fig. 12 shows the two-lune arch of minimum thickness at the point of collapse, with hinges at P, Q and R. The corresponding collapse state for the complete dome is shown in Fig. 13, and it will be seen that the portion PP near the crown of the dome remains coherent (cf. Fig. 15), whereas the hinges at P, Q and R ensure that the dome does indeed divide itself into lunes, so that no hoop stress can be generated. By contrast, although the analysis has been carried out for thin slices, the portion PP of the dome can develop the usual meridional and hoop stresses, and in fact must do so if the dome has an eye, or, as is the case of St. Peter's, if it carries a lantern. The example of the Pantheon, which has an eye, is discussed later.

58. For the complete uniform dome, having no eye or lantern, the minimum

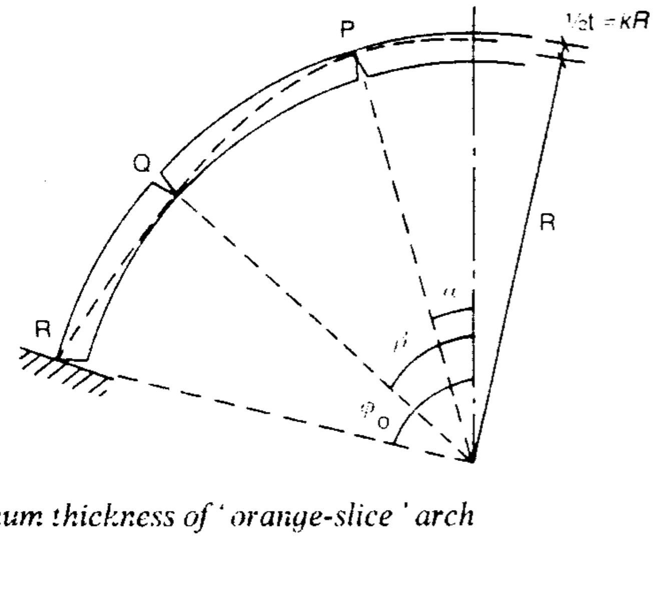


Fig. 12. Minimum thickness of 'orange-slice' arch

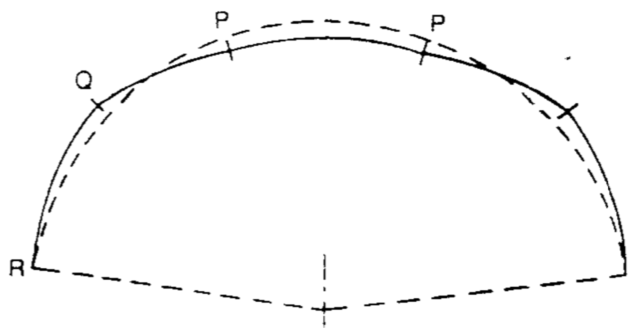


Fig. 13. Collapse state corresponding to Fig. 12

thickness required for stability falls, as might be expected, as the cut-off angle ϕ_o is reduced (Fig. 12). At $\phi_o = 51{\cdot}8°$ points P and Q coincide and the thickness falls to zero, corresponding to the well known fact that both meridional and hoop stresses in thin shell domes are compressive at the crown and for considerable angles away from the crown. Fig. 14 shows the minimum thickness required for stability as a function of the cut-off angle ϕ_o. For angles ϕ_o less than $51{\cdot}8°$ a spherical masonry dome can, in theory, be built with a vanishingly small thickness. A hemispherical dome requires a minimum thickness of just over 2% of the diameter. Practical domes have, of course, thicknesses in excess of this minimum and, in addition, may not approximate to full hemispheres. St. Peter's dome has a thickness to diameter ratio of about 5%, whereas the Pantheon in Rome has a ratio of about 3·3%. The expected crack pattern of such domes may be investigated.

59. From the previous examples of the simple arch and of the cross-vault, it is to be supposed that the supporting structure of the dome will give way slightly under the thrust generated by the dome itself. The geometrical mismatch must be accommodated as usual by cracking, and the pattern for a complete dome exhibiting symmetrical uniform defects is sketched schematically in Fig. 15. From the base of the dome, meridional cracks will rise towards the crown, but there will be a coherent central portion (cf. the region PP in Fig. 13). Below this level the movement outward of the supports will ensure the development of the lunes on which the analysis can be based.

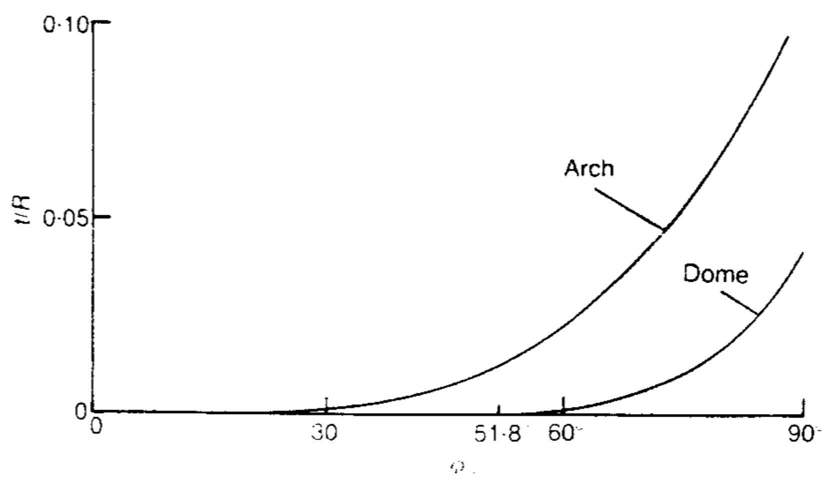


Fig. 14. Minimum thicknesses of circular arches and spherical domes

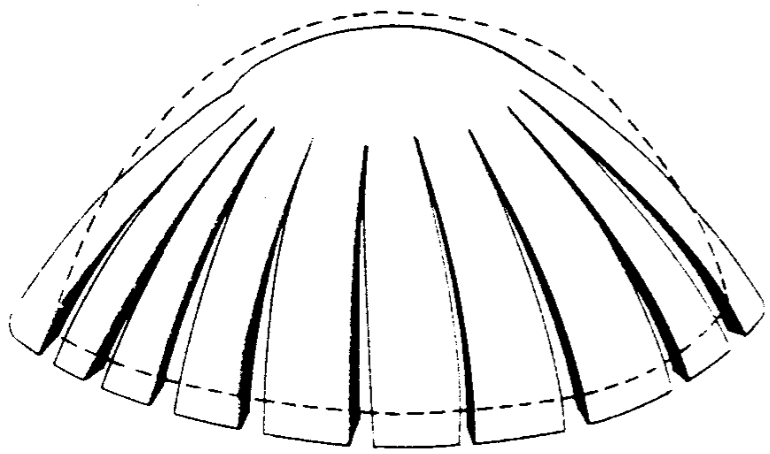


Fig. 15. Schematic illustration of cracking in dome due to increase of span

60. A partial cross-section is shown in Fig. 16. If this Figure is compared with Fig. 12, it can be seen that the hinge at Q is shown at the base. A straightforward intrinsic equation, albeit of some complexity, can be written from which the angle α (dividing the solid cap of the dome from the lunes) can be determined in terms of the angle β and the thickness $t = 2kR$, where R is the radius of the dome. For cut-off angles ϕ_0 greater than about 65°, but depending very slightly on the thickness t, the thrust line becomes tangent to the intrados at Q, and the hinge no longer forms at the base of the lune (cf. Fig. 6(*a*) for the two-dimensional arch).

61. The complex equation can be solved numerically, and results are plotted in Fig. 17, from which the value of α defining the uncracked portion of the dome can be read for any value of the non-dimensional thickness $k(= t/2R)$. For St. Peter's, for example, for which $\phi_0 = 90°$ and $k \simeq 0{\cdot}05$, the topmost curve gives $\alpha = 24°$; the disturbing meridional cracks which led to Poleni's report should run up some 66° from the base of the dome, and this value accords well with Poleni's drawings.

62. Figure 17 has been calculated for the uniform thickness spherical dome, and can only approximate to the dome of St. Peter's. However, the results seem applicable, and this is supported by a more detailed analysis of the Pantheon. This Roman concrete structure is shown in cross-section in Fig. 18. The dome has an open eye of about 9 m dia., and the dome itself can be idealized as spherical with a mean radius of 22·4 m and thickness 1·50 m. Thus k in Fig. 17 is about 0·033 and, if the dome were truly hemispherical, α would be read off as about 28°.

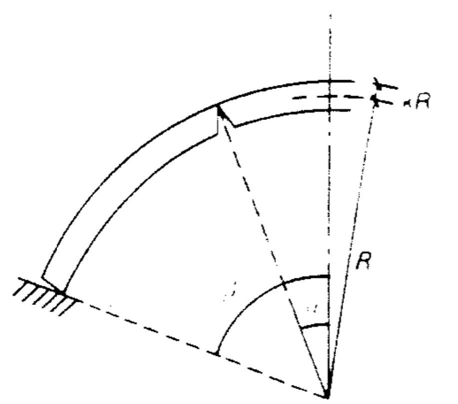


Fig. 16. Cross-section of dome corresponding to Fig. 15

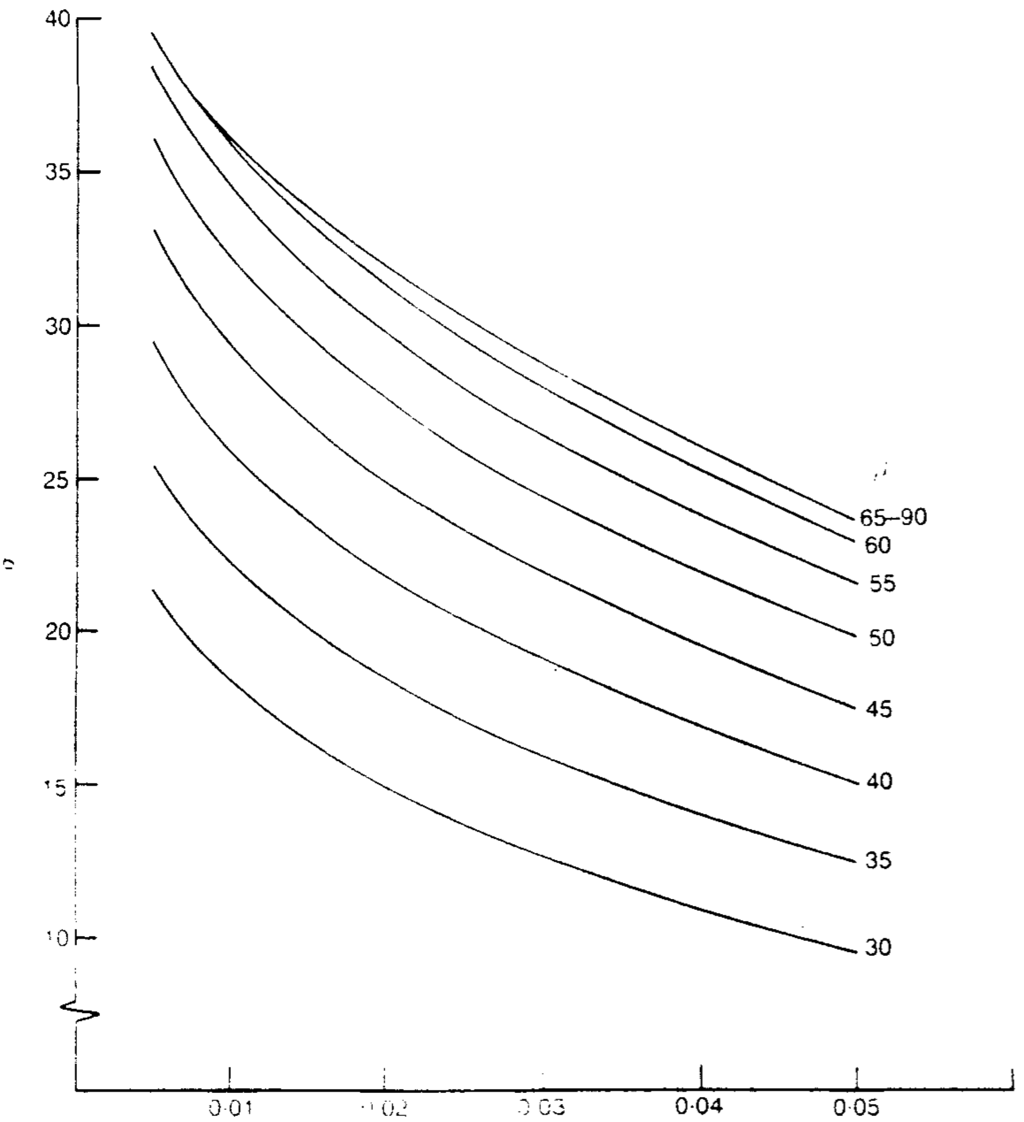


Fig. 17. Extent of cracking of spherical dome (cf. Figs 15 and 16)

63. This value of 28° accords well with the value of about 26° found by constructing the thrust line on the drawing board. Fig. 19 shows (solid line) the thrust line constructed down to about 60° from the crown; the cut-off angle of 60° corresponds to the greatly increased thickness of the construction, Fig. 18, where the dome runs into the circular supporting wall.

64. Figure 19 shows (broken line) a further thrust line for the dome with an open eye. The extent of cracking ($\alpha = 26°$) is almost unchanged, and Terenzio's report of 50 years ago,[10] as noted by Mark and Hutchinson,[11] states that the meridional cracks extended to an average of about 57° above the springing (i.e. $\alpha = 23°$). Thus the extent of cracking in the real dome is predicted quite well both by the study of a complete dome without an eye, and also by the calculations recorded in Fig. 17.

65. The thrust lines of Fig. 19 correspond to the forces in the lunes defined by the extent of the cracking, that is for angles between 26° and the cut-off at 60°. For smaller angles the thrust line for the solid dome represents one of the possibilities

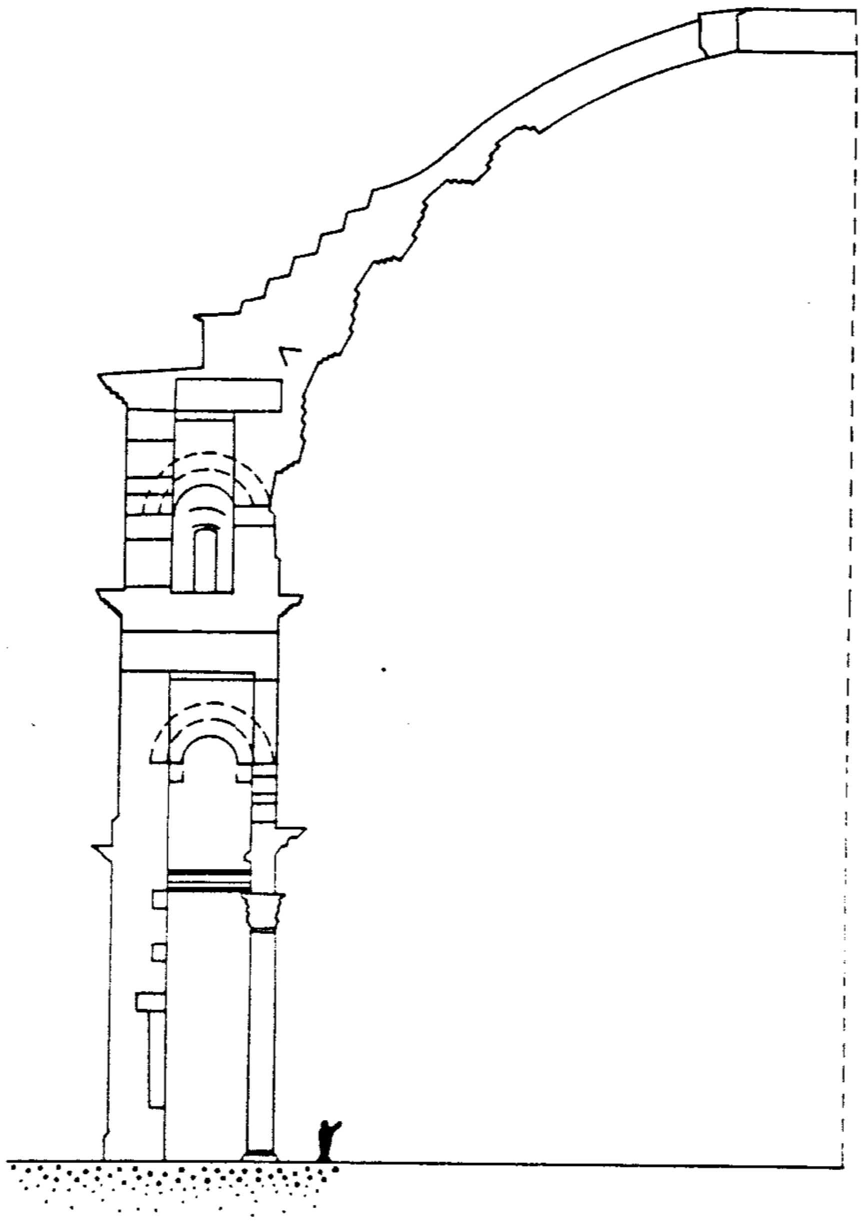


Fig. 18. Cross-section of the Pantheon, Rome

for equilibrium, but, as has been noted, the cap of the dome remains in a coherent uncracked state (cf. Fig. 15). The broken line of thrust for the dome with an eye is meaningless for the cap portion having meridional angles smaller than 26°. The meridional stresses must die away to zero at the radius of the eye, and the coherent central cap can only be maintained in equilibrium if compressive hoop stresses are developed. However, such 'membrane' equilibrium is indeed possible for the cap; the broken line of thrust in Fig. 19 is valid below the cap for the cracked lune.

66. In the whole of this discussion of the cracking of domes it has been assumed that the outward movement of the periphery was uniform. In practice, of course, the boundary movements will be highly non-uniform. An idealized dome of the usual rigid/unilateral material is shown in Fig. 20 for which three quarters of the periphery has stood firm, while one quarter has suffered a small outward displacement. The slight change in geometry will allow lunes to develop in one

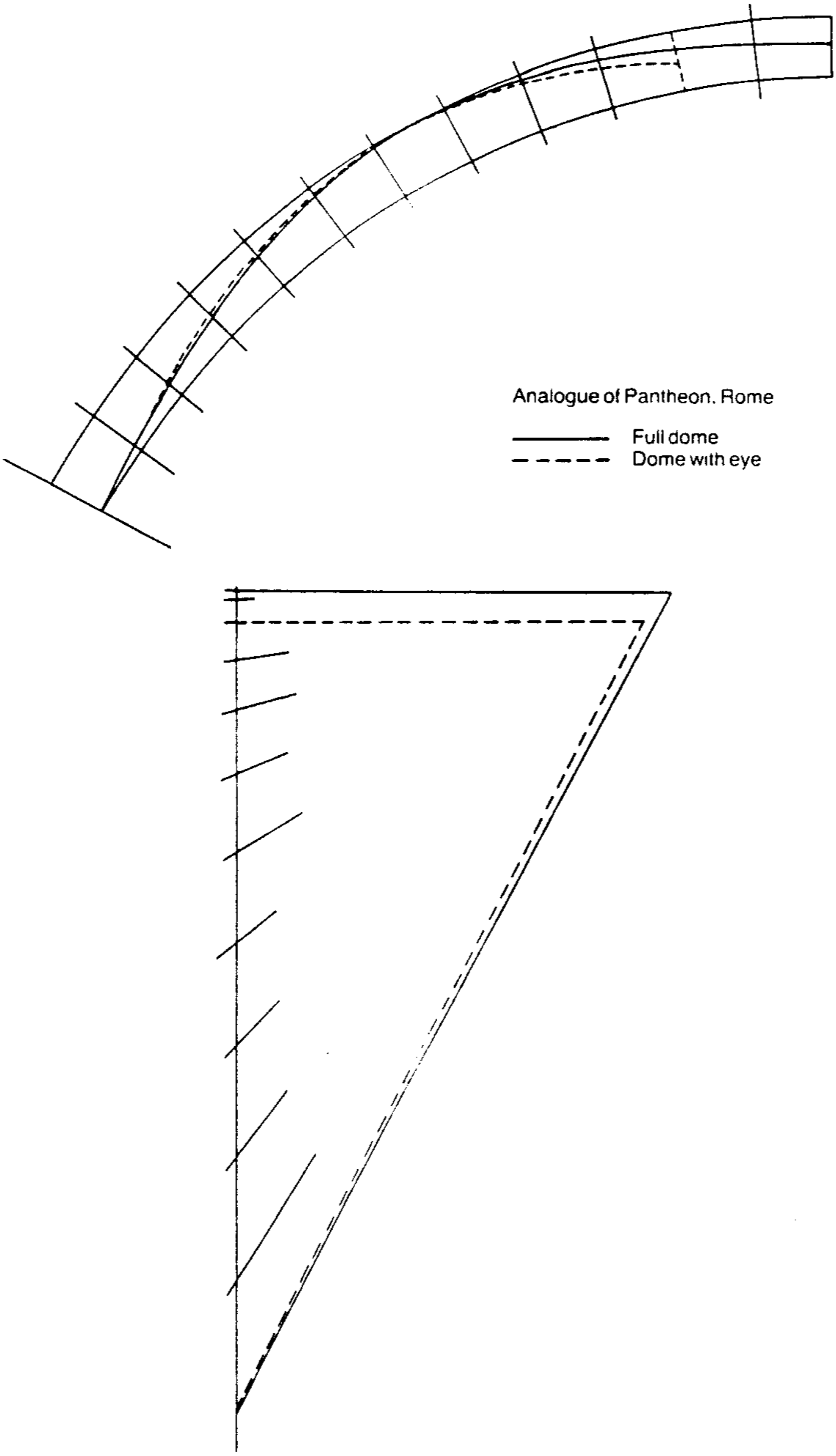


Fig. 19. Thrust lines for analogue of the Pantheon

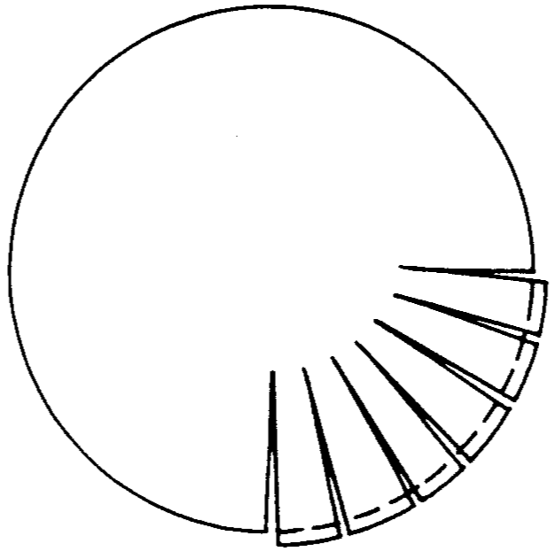


Fig. 20. Partial cracking of a dome (schematic)

quarter of the dome (in practice, of course, in an irregular pattern), without impairing the stability of the three quarters of the dome which remain intact.

67. The crack pattern of Fig. 21 for the dome of Santa Maria del Fiore in Florence[12,13] indeed indicates where movements have taken place. Brunelleschi's octagonal segmental dome is buttressed on the north, south and east by domed apses, and on the west by the nave of the cathedral. Between these four cardinal points, however, the dome is less strongly supported, and there has been outward movement at the periphery. The cracks point towards the regions where movement has occurred.

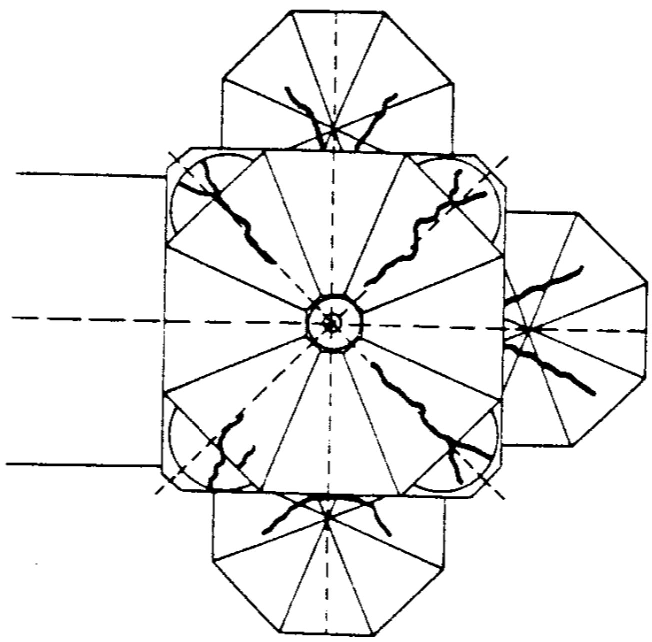


Fig. 21. Cracking in the dome of Santa Maria del Fiore, Florence (Blasi and Ceccotti).

Conclusion

68. Equilibrium provides a powerful key to unlock the secrets of structural behaviour. Despite the deliberate rejection of an 'elastic' analysis which might lead to the calculation of the actual structural state, the use of statics alone can furnish, on the one hand, good estimates of structural quantities, and on the other an understanding of the meaning of crack patterns in masonry. Poleni was able, from a measure of the force with which his experimental chain pulled at its abutments, to estimate the thrust which the dome imposed on its supporting drum. He was able to account for the observed pattern of cracks, and he drew the correct conclusion that the cracks did not affect the stability of the dome. Above all, Poleni knew that it was essential to ensure a correct geometry in masonry construction. The medieval lodges knew this too, without the support of Poleni's rational mechanics. The medieval rules, which can be traced back to Roman practice, were arrived at by trial and error, but were directed precisely to relating the proportions of the different parts of a structure. As such, they were essentially the right kind of rule.

Acknowledgement

69. Part of the material in this paper was first presented in a course of lectures on masonry structures given in May 1987 at the University of Florence (Dipartimento di Costruzioni).

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