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ON THE PROPAGATION OF LIGHT IN CRYSTALLIZED MEDIA.

In a former paper* I endeavoured to determine in what way a plane wave would be modified when transmitted from one non-crystallized medium to another; founding the investigation on this principle: In whatever manner the elements of any material system may act upon each other, if all the internal forces be multiplied by the elements of their respective directions, the total sums for any assigned portion of the mass will always be the exact differential of some function. This principle requires a slight limitation, and when the necessary limitation is introduced, appears to possess very great generality. I shall here endeavour to apply the same principle to crystallized bodies, and shall likewise introduce the consideration of the effects of extraneous pressures, which had been omitted in the former communication. Our problem thus becomes very complicated, as the function due to the internal forces, even when there are no extraneous pressures, contains twenty-one coefficients. But with these pressures we are obliged to introduce six additional coefficients; so that without some limitation, it appears quite hopeless thence to deduce any consequences which could have the least chance of a physical application. The absolute necessity of introducing some arbitrary restrictions, and the desire that their number should be as small as possible, induced me to examine how far our function would be limited by confining ourselves to the consideration of those media only in which the directions of the transverse vibrations shall always be *accurately* in the front of the wave. This fundamental principle of Fresnel's Theory gives fourteen relations between the twenty-one constants originally entering into our function; and it seems worthy of remark, that when there are no extraneous pressures, the directions of polarization and the wave-velocities given by our theory, when thus limited, are identical with those assigned by Fresnel's general construction for biaxal crystals; provided we suppose the actual direction of disturbance in the particles

* Supra, p. 243.

of the medium is *parallel* to the plane of polarization, agreeably to the supposition first advanced by M. Cauchy.

If we admit the existence of extraneous pressures, it will be necessary in addition to the single restriction before noticed, to suppose that for three plane waves parallel to three orthogonal sections of our medium, and which may be denominated principal sections, the wave-velocities shall be the same for any two of the three waves whose fronts are parallel to these sections, provided the direction of the corresponding disturbances are parallel to the line of their intersection. With this additional supposition, the directions of the actual disturbances by which any plane wave will propagate itself without subdivision, and the wave-velocities, agree exactly with those given by Fresnel, supposing, with him, that these directions are *perpendicular* to the plane of polarization. The last, or Fresnel's hypothesis, was adopted in our former paper. But as that paper relates merely to the intensities of the waves reflected and refracted at the surface of separation of two media, and as these intensities may depend upon physical circumstances, the consideration of which was not introduced into our former investigations, it seems right, in the present paper, considering the actual situation of the theory of light, when the partial differential equations on which the determination of the motion of the luminiferous ether depends are yet to discover, to state fairly the results of both hypotheses.

It is hoped the analysis employed on the present occasion will be found sufficiently simple as a method has here been given of passing immediately and without calculation from the function due to the internal forces of our medium to the equation of an ellipsoidal surface, of which the semi-axes represent in magnitude the reciprocals of the three wave-velocities, and in direction the directions of the three corresponding disturbances by which a wave can propagate itself in one medium without subdivision. This surface, which may be properly styled the ellipsoid of elasticity, must not be confounded with the one whose section by a plane parallel to the wave's front gives the reciprocals of the wave-velocities, and the corresponding directions of polarization. The two surfaces have only this section in common*, and a very simple application of our theory would shew that no force perpendicular to the wave's front is rejected, as in the ordinary one, but that the force in question is absolutely null†.

Let us conceive a system composed of an immense number of particles mutually acting on each other, and moreover subjected to the influence of extraneous pressures. Then if x, y, z are the co-ordinates of any particle of this system in its primitive state, (that of equilibrium under pressure for example), the co-ordinates of the same particle at the end of the time t will become x', y', z', where x', y', z' are functions of x, y, z and t. If now we consider an element of this medium, of which the primitive form is that of a rectangular parallelopiped, whose sides are dx, dy, dz, this element in its new state will assume the form of an oblique-angled parallelopiped, the lengths of the three edges being (dx'), (dy'), (dz'), these edges being composed of the same particles which formed the three edges dx, dy, dz in the primitive state of the element. Then will

$$\left.\begin{aligned}(dx')^2 &= \left\{\left(\frac{dx'}{dx}\right)^2 + \left(\frac{dy'}{dx}\right)^2 + \left(\frac{dz'}{dx}\right)^2\right\} dx^2 = a^2 dx^2\\ (dy')^2 &= \left\{\left(\frac{dx'}{dy}\right)^2 + \left(\frac{dy'}{dy}\right)^2 + \left(\frac{dz'}{dy}\right)^2\right\} dy^2 = b^2 dy^2\\ (dz')^2 &= \left\{\left(\frac{dx'}{dz}\right)^2 + \left(\frac{dy'}{dz}\right)^2 + \left(\frac{dz'}{dz}\right)^2\right\} dz^2 = c^2 dz^2\end{aligned}\right\} \text{suppose.}$$

Again, let

$$\alpha = \cos\angle\binom{dy'}{dz'} = \frac{\dfrac{dx'}{dy}\dfrac{dx'}{dz} + \dfrac{dy'}{dy}\dfrac{dy'}{dz} + \dfrac{dz'}{dy}\dfrac{dz'}{dz}}{\sqrt{\left\{\left(\dfrac{dx'}{dy}\right)^2+\left(\dfrac{dy'}{dy}\right)^2+\left(\dfrac{dz'}{dy}\right)^2\right\}\left\{\left(\dfrac{dx'}{dz}\right)^2+\left(\dfrac{dy'}{dz}\right)^2+\left(\dfrac{dz'}{dz}\right)^2\right\}}},$$

* [It will be seen that this remark is not strictly correct, as the surface must necessarily have another common plane section.]

† [Referring to the values of u, v, w given in p. 301, we see that, since the direction of vibration is supposed to be in the front of the wave, we have

$$au + bv + cw = 0.$$

But the force perpendicular to the wave's front is $a\dfrac{d^2u}{dt^2} + b\dfrac{d^2v}{dt^2} + w\dfrac{d^2w}{dt^2}$, which is equal to $s^2(au + bv + cw)$, and is therefore null.]

$$\beta = \cos\angle\binom{dx'}{dz'} = \frac{\frac{dx'}{dx}\frac{dx'}{dz} + \frac{dy'}{dx}\frac{dy'}{dz} + \frac{dz'}{dx}\frac{dz'}{dz}}{\sqrt{\left\{\left(\frac{dx'}{dx}\right)^2 + \left(\frac{dy'}{dx}\right)^2 + \left(\frac{dz'}{dx}\right)^2\right\}\left\{\left(\frac{dx'}{dz}\right)^2 + \left(\frac{dy'}{dz}\right)^2 + \left(\frac{dz'}{dz}\right)^2\right\}}},$$

$$\gamma = \cos\angle\binom{dx'}{dy'} = \frac{\frac{dx'}{dx}\frac{dx'}{dy} + \frac{dy'}{dx}\frac{dy'}{dy} + \frac{dz'}{dx}\frac{dz'}{dy}}{\sqrt{\left\{\left(\frac{dx'}{dx}\right)^2 + \left(\frac{dy'}{dx}\right)^2 + \left(\frac{dz'}{dx}\right)^2\right\}\left\{\left(\frac{dx'}{dy}\right)^2 + \left(\frac{dy'}{dy}\right)^2 + \left(\frac{dz'}{dy}\right)^2\right\}}};$$

or we may write

$$\alpha' = bc\alpha = \frac{dx'}{dy}\frac{dx'}{dz} + \frac{dy'}{dy}\frac{dy'}{dz} + \frac{dz'}{dy}\frac{dz'}{dz},$$

$$\beta' = ac\beta = \frac{dx'}{dx}\frac{dx'}{dz} + \frac{dy'}{dx}\frac{dy'}{dz} + \frac{dz'}{dx}\frac{dz'}{dz},$$

$$\gamma' = ab\gamma = \frac{dx'}{dx}\frac{dx'}{dy} + \frac{dy'}{dx}\frac{dy'}{dy} + \frac{dz'}{dx}\frac{dz'}{dy}.$$

Suppose now, as in a former paper, that $\phi\, dx\, dy\, dz$ is the function due to the mutual actions of the particles which compose the element whose primitive volume $= dx\, dy\, dz$. Since ϕ must remain the same, when the sides (dx'), (dy'), (dz') and the cosines α, β, γ of the angles of the elementary oblique-angled parallelopiped remain unchanged, its most general form must be

$$\phi = \text{function}\ (a, b, c, \alpha, \beta, \gamma),$$

or since a, b, and c are necessarily positive, also

$$\alpha' = bc\alpha,\quad \beta' = ac\beta,\quad \text{and}\quad \gamma' = ab\gamma,$$

we may write $\qquad \phi = f(a^2, b^2, c^2, \alpha', \beta', \gamma')$..................(1).

This expression is the equivalent of the one immediately preceding, and is here adopted for the sake of introducing greater symmetry into our formulæ.

We will in the first place suppose that ϕ is symmetrical with regard to three planes at right angles to each other, which we shall take as the co-ordinate planes. The condition of symmetry with respect to the plane (yz), will require ϕ to remain unchanged, when we change

$$\left.\begin{matrix} x \\ x' \end{matrix}\right\} \text{ into } \left\{\begin{matrix} -x \\ -x'. \end{matrix}\right.$$

But thus a^2, b^2, c^2 and α' evidently remain unaltered; moreover

$$\left.\begin{matrix} \beta' \\ \gamma' \end{matrix}\right\} \text{ become } \left\{\begin{matrix} -\beta' \\ -\gamma'. \end{matrix}\right.$$

Hence we get

$$\phi = f(a^2, b^2, c^2, \alpha'^2, \beta'^2, \gamma'^2).$$

Applying the like reasoning to the other co-ordinate planes, we see that the ultimate result will be

$$\phi = f(a^2, b^2, c^2, \alpha'^2, \beta'^2, \gamma'^2) \text{ ....(2).}$$

The foregoing values are perfectly general, whatever the disturbance may be; but if we consider this disturbance as very small, we may make

$$x' = x + u,$$

$$y' = y + v,$$

$$z' = z + w,$$

u, v, and w being very small functions of x, y, z, and t of the first order. Then by substitution we get

$$\left.\begin{aligned} a^2 &= 1 + 2\frac{du}{dx} + \left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2 = 1 + s_1 \\ b^2 &= 1 + 2\frac{dv}{dy} + \left(\frac{du}{dy}\right)^2 + \left(\frac{dv}{dy}\right)^2 + \left(\frac{dw}{dy}\right)^2 = 1 + s_2 \\ c^2 &= 1 + 2\frac{dw}{dz} + \left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2 + \left(\frac{dw}{dz}\right)^2 = 1 + s_3 \end{aligned}\right\} \text{ suppose ...(3),}$$

$$\alpha' = \frac{dv}{dz} + \frac{dw}{dy} + \frac{du}{dy}\frac{du}{dz} + \frac{dv}{dy}\frac{dv}{dz} + \frac{dw}{dy}\frac{dw}{dz},$$

$$\beta' = \frac{du}{dz} + \frac{dw}{dx} + \frac{du}{dx}\frac{du}{dz} + \frac{dv}{dx}\frac{dv}{dz} + \frac{dw}{dx}\frac{dw}{dz},$$

$$\gamma' = \frac{du}{dy} + \frac{dv}{dx} + \frac{du}{dx}\frac{du}{dy} + \frac{dv}{dx}\frac{dv}{dy} + \frac{dw}{dx}\frac{dw}{dy};$$

we thus see that $s_1, s_2, s_3, \alpha', \beta', \gamma'$, are very small quantities of the first order, and that the general formula (1) by substituting the preceding values would take the form

$$\phi = \text{function } (s_1, s_2, s_3, \alpha', \beta', \gamma'),$$

which may be expanded in a very convergent series of the form

$$\phi = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \&c.:$$

ϕ_0, ϕ_1, ϕ_2, &c. being homogeneous functions of $s_1, s_2, s_3\ \alpha', \beta', \gamma'$, of the degrees 0, 1, 2, 3, &c. each of which is very great compared with the next following one.

But ϕ_0 being constant, if $\rho =$ the primitive density of the element, the general formula of Dynamics will give

$$\iiint \rho\, dx dy dz \left\{\frac{d^2u}{dt^2}\delta u + \frac{d^2v}{dt^2}\delta v + \frac{d^2w}{dt^2}\delta w\right\} = \iiint dx dy dz (\delta\phi_1 + \delta\phi_2 + \&c.).$$

If there were no extraneous pressures, the supposition that the primitive state was one of equilibrium would require $\phi_1 = 0$, as was observed in a former paper; but this is not the case if we introduce the consideration of extraneous pressures. However, as in the first case, the terms ϕ_3, ϕ_4, &c. will be insensible and the preceding formula may be written

$$\iiint \rho\, dx dy dz \left\{\frac{d^2u}{dt^2}\delta u + \frac{d^2v}{dt^2}\delta v + \frac{d^2w}{dt^2}\delta w\right\} = \iiint dx dy dz\ (\delta\phi_1 + \delta\phi_2).$$

Supposing ρ the primitive density constant, the most general form of ϕ_1 will be

$$\phi_1 = -\frac{1}{2}(As_1 + Bs_2 + Cs_3 + 2D\alpha' + 2E\beta' + 2F\gamma'),$$

A, B, C, D, E, and F being constant quantities.

In like manner the most general form of ϕ_2 will contain twenty-one coefficients. But if we first employ the more particular value (2), we shall get

$$-2\phi_1 = As_1 + Bs_2 + Cs_3$$

$$-2\phi_2 = Gs_1^2 + Hs_2^2 + Is_3^2 + 2Ps_2s_3 + 2Qs_1s_3 + 2Rs_1s_2$$

$$+ L\alpha'^2 + M\beta'^2 + N\gamma'^2.$$

Or by substituting for s_1, s_2, s_3, α', β', γ' their values, given by system (3), continuing to neglect quantities of the third order, we get

$$-2\phi = -2\phi_1 - 2\phi_2$$

$$= 2A\frac{du}{dx} + 2B\frac{dv}{dy} + 2C\frac{dw}{dz}$$

$$+ A\left\{\left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right\}$$

$$+ B\left\{\left(\frac{du}{dy}\right)^2 + \left(\frac{dv}{dy}\right)^2 + \left(\frac{dw}{dy}\right)^2\right\}$$

$$+ C\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2 + \left(\frac{dw}{dz}\right)^2\right\}$$

$$+ G\left(\frac{du}{dx}\right)^2 + H\left(\frac{dv}{dy}\right)^2 + I\left(\frac{dw}{dz}\right)^2 + 2P\frac{dv}{dy}\frac{dw}{dz} + 2Q\frac{du}{dx}\frac{dw}{dz} + 2R\frac{du}{dx}\frac{dv}{dy}$$

$$+ L\left(\frac{dv}{dz} + \frac{dw}{dy}\right)^2 + M\left(\frac{du}{dz} + \frac{dw}{dx}\right)^2 + N\left(\frac{du}{dy} + \frac{dv}{dx}\right)^2 \ldots\ldots(4).$$

Having thus the form of the function due to the internal actions of the particles, we have merely to substitute it in the general formula of Dynamics, and to effect the integrations by parts, agreeably to the method of Lagrange. Thus,

$$\iiint dx\,dy\,dz\,\delta\phi =$$

$$-\iint dy\,dz\left\{A\delta u + A\left(\frac{du}{dx}\delta u + \frac{dv}{dx}\delta v + \frac{dw}{dx}\delta w\right) + \left(G\frac{du}{dx} + R\frac{dv}{dy} + Q\frac{dw}{dz}\right)\delta u + M\left(\frac{du}{dz} + \frac{dw}{dx}\right)\delta w + N\left(\frac{du}{dy} + \frac{dv}{dx}\right)\delta v\right\}$$

$$-\iint dx\,dz\left\{B\delta v + B\left(\frac{du}{dy}\delta u + \frac{dv}{dy}\delta v + \frac{dw}{dy}\delta w\right) + \left(R\frac{du}{dx} + H\frac{dv}{dy} + P\frac{dw}{dz}\right)\delta v + L\left(\frac{dv}{dz} + \frac{dw}{dy}\right)\delta w + N\left(\frac{du}{dy} + \frac{dv}{dx}\right)\delta u\right\}$$

$$-\iint dx\,dy\left\{C\delta w + C\left(\frac{du}{dz}\delta u + \frac{dv}{dz}\delta v + \frac{dw}{dz}\delta w\right) + \left(Q\frac{du}{dx} + P\frac{dv}{dy} + I\frac{dw}{dz}\right)\delta w + L\left(\frac{dv}{dz} + \frac{dw}{dy}\right)\delta v + M\left(\frac{du}{dz} + \frac{dw}{dx}\right)\delta u\right\}$$

$$+\iiint dx\,dy\,dz\,\delta u\left\{(G+A)\frac{d^2u}{dx^2} + (N+B)\frac{d^2u}{dy^2} + (M+C)\frac{d^2u}{dz^2} + (R+N)\frac{d^2v}{dx\,dy} + (Q+M)\frac{d^2w}{dx\,dz}\right\}$$

$$+\iiint dx\,dy\,dz\,\delta v\left\{(N+A)\frac{d^2v}{dx^2} + (H+B)\frac{d^2v}{dy^2} + (L+C)\frac{d^2v}{dz^2} + (N+R)\frac{d^2u}{dx\,dy} + (P+L)\frac{d^2w}{dy\,dz}\right\}$$

$$+\iiint dx\,dy\,dz\,\delta w\left\{(M+A)\frac{d^2w}{dx^2} + (L+B)\frac{d^2w}{dy^2} + (I+C)\frac{d^2w}{dz^2} + (M+Q)\frac{d^2u}{dx\,dz} + (L+P)\frac{d^2v}{dy\,dz}\right\}.$$

Neglecting the double integrals which relate to the extreme boundaries of the medium, and which we will suppose situated at an infinite distance, we get for the general equations of motion,

$$\left.\begin{aligned}
\rho\frac{d^2u}{dt^2} &= (G+A)\frac{d^2u}{dx^2} + (N+B)\frac{d^2u}{dy^2} + (M+C)\frac{d^2u}{dz^2} + (R+N)\frac{d^2v}{dx\,dy} + (Q+M)\frac{d^2w}{dx\,dz}\\
\rho\frac{d^2v}{dt^2} &= (N+A)\frac{d^2v}{dx^2} + (H+B)\frac{d^2v}{dy^2} + (L+C)\frac{d^2v}{dz^2} + (N+R)\frac{d^2u}{dx\,dy} + (P+L)\frac{d^2w}{dy\,dz}\\
\rho\frac{d^2w}{dt^2} &= (M+A)\frac{d^2w}{dx^2} + (L+B)\frac{d^2w}{dy^2} + (I+C)\frac{d^2w}{dz^2} + (M+Q)\frac{d^2u}{dx\,dz} + (L+P)\frac{d^2v}{dy\,dz}
\end{aligned}\right\}\ \text{.........}\ (5).$$

If now in our indefinitely extended medium we wish to determine the laws of propagation of plane waves, we must take, to satisfy the last equations,

$$u = \alpha f(ax + by + cz + et),$$
$$v = \beta f(ax + by + cz + et),$$
$$w = \gamma f(ax + by + cz + et);$$

a, b, and c being the cosines of the angles which a normal to the wave's front makes with the co-ordinate axes, α, β, γ constant coefficients, and e the velocity of transmission of a wave perpendicular to its own front, and taken with a contrary sign.

Substituting these values in the equations (5), and making to abridge

$$A' = (G + A)\,a^2 + (N + B)\,b^2 + (M + C)\,c^2,$$
$$B' = (N + A)\,a^2 + (H + B)\,b^2 + (L + C)\,c^2,$$
$$C' = (M + A)\,a^2 + (L + B)\,b^2 + (I + C)\,c^2;$$
$$D' = (L + P)\,bc,$$
$$E' = (M + Q)\,ac,$$
$$F' = (N + R)\,ab;$$

we get

$$\left.\begin{array}{l} 0 = (A' - \rho e^2)\,\alpha + F'\beta + E'\gamma \\ 0 = F'\alpha + (B' - \rho e^2)\,\beta + D'\gamma \\ 0 = E'\alpha + D'\beta + (C' - \rho e^2)\,\gamma \end{array}\right\}\ldots(6).$$

These last equations will serve to determine three values of ρ^2, and three corresponding ratios of the quantities α, β, γ; and hence we know the directions of the disturbance by which a plane wave will propagate itself without subdivision, and also the corresponding velocities of propagation. From the form of the equations (6), it is well known, that if we conceive an ellipsoid whose equation is

$$1 = A'x^2 + B'y^2 + C'z^2 + 2D'yz + 2E'xz + 2F'xy^{*}\ldots\ldots(7),$$

* If we reflect on the connexion of the operations by which we pass from the function (4) to the equation (7), it will be easy to perceive that the right side of the equation (7) may always be immediately deduced from that portion of the function

and represent its three semi-axes by r', r'', and r''', the directions of these axes will be the required directions of the disturbance, and the corresponding velocities of propagation will be given by

$$\rho s^2 = \frac{1}{r^2}.$$

Fresnel supposes those vibrations of the particles of the luminiferous ether which affect the eye, to be *accurately* in the front of the wave.

Let us therefore investigate the relation which must exist between our coefficients, in order to satisfy this condition for two out of our three waves, the remaining one in consequence being necessarily propagated by normal vibrations

For this we may remark, that the equation of a plane parallel to the wave's front is

$$0 = ax' + by' + cz' \ldots\ldots (a)$$

If therefore we make

$$x = x' + a\lambda,$$
$$y = y' + b\lambda,$$
$$z = z' + c\lambda,$$

and substitute these values in the equation (7) of the ellipsoid; restoring the values of

$$A', B', C', D', E', F',$$

the odd powers of λ ought to disappear in consequence of the equation (a), whatever may be the position of the wave's front. We thus get

$$G = H = I = \mu \text{ suppose,}$$

and

$$P = \mu - 2L,$$
$$Q = \mu - 2M,$$
$$R = \mu - 2N.$$

which is of the second degree, by changing u, v, and w into x, y, and z. Also $\frac{d}{dx}$, $\frac{d}{dy}$ and $\frac{d}{dz}$ into a, b, c.

This remark will be of use to us afterwards, when we come to consider the most general form of the function due to the internal actions.

In fact, if we substitute these values in the function (4), there will result

$$
\begin{aligned}
-2\phi &= -2\phi_1 - 2\phi_2 \\
&= 2A\frac{du}{dx} + 2B\frac{dv}{dy} + 2C\frac{dw}{dz} \\
&+ A\left\{\left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right\} \\
&+ B\left\{\left(\frac{du}{dy}\right)^2 + \left(\frac{dv}{dy}\right)^2 + \left(\frac{dw}{dy}\right)^2\right\} \\
&+ C\left\{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2 + \left(\frac{dw}{dz}\right)^2\right\} \text{ ....}(A), \\
&+ \mu\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)^2 \\
&+ L\left\{\left(\frac{dv}{dz} + \frac{dw}{dy}\right)^2 - 4\frac{dv}{dy}\frac{dw}{dz}\right\} \\
&+ M\left\{\left(\frac{du}{dz} + \frac{dw}{dx}\right)^2 - 4\frac{du}{dx}\frac{dw}{dz}\right\} \\
&+ N\left\{\left(\frac{du}{dy} + \frac{dv}{dx}\right)^2 - 4\frac{du}{dy}\frac{dv}{dx}\right\}
\end{aligned}
$$

which, when $0 = A$, $0 = B$, $0 = C$, reduces to the last four lines.

Making the same substitution in the equation (7), we get

$$
\left.
\begin{aligned}
1 = \mu\,(ax + by + cz)^2 \\
+ (Aa^2 + Bb^2 + Cc^2)\,(x^2 + y^2 + z^2) \\
+ L\,(cy - bz)^2 + M\,(az - cx)^2 + N\,(bx - ay)^2
\end{aligned}
\right\} \text{......}(8).
$$

Let us in the first place suppose the system free from all extraneous pressure.

Then $A = 0,\ \ B = 0,\ \ C = 0,$

and the above equation, combined with that of a plane parallel to the wave's front, will give

$$0 = ax + by + cz \text{ .... } (9),$$

$$1 = L\,(cy - bz)^2 + M\,(az - cx)^2 + N\,(bx - ay)^2,$$

the equations of an infinite number of ellipses which, in general, do not belong to the same curve surface. If, however, we cause each ellipsis to turn 90° in its own plane, the whole system will belong to an ellipsoid, as may be thus shewn: Let (xyz) be the co-ordinates of any point p in its original position, and $(x'y'z')$ the co-ordinates of the point p' which would coincide with p when the ellipsis is turned 90° in its own plane. Then

$$x^2+y^2+z^2=x'^2+y'^2+z'^2,$$

since the distance from the origin O is unaltered,

$$0=ax'+by'+cz', \text{ since the plane is the same,}$$

$$0=xx'+yy'+zz', \text{ since } pOp'=90^\circ.$$

The two last equations give

$$\frac{x'}{cy-bz}=\frac{y'}{az-cx}=\frac{z'}{bx-ay}=w \text{ suppose.}$$

Hence the last of the equations (9) becomes

$$\omega^2=Lx'^2+My'^2+Nz'^2.$$

But

$$x'^2+y'^2+z'^2=\omega^2\{(cy-bz)^2+(az-cx)^2+(bx-ay)^2\}$$
$$=\omega^2\{(b^2+a^2)z^2+(c^2+a^2)y^2+(b^2+c^2)x^2-2(bcyz+abxy+acxz\}$$
$$=\omega^2\{(a^2+b^2+c^2)(x^2+y^2+z^2)-(ax+by+cz)^2\}$$
$$=\omega^2(x^2+y^2+z^2)=x^2+y^2+z^2.$$

Therefore $$\omega^2=1,$$

and our equation finally becomes

$$1=Lx'^2+My'^2+Nz'^2\dots\dots\dots\dots\dots\dots(10).$$

We thus see that if we conceive a section made in the ellipsoid to which the equation (10) belongs, by a plane passing through its centre and parallel to the wave's front, this section, when turned 90 degrees in its own plane, will coincide with a similar section of the ellipsoid to which the equation (8) belongs, and which gives the directions of the disturbance that will cause a plane wave to propagate itself without subdivision, and the velocity of propagation parallel to its own front. The change of position here made in the elliptical section, is evidently equivalent to supposing the actual disturbances of the ethereal particles to be parallel to the plane usually denominated the *plane of polarization*.

This hypothesis, at first advanced by M. Cauchy, has since been adopted by several philosophers; and it seems worthy of remark, that if we suppose an elastic medium free from all extraneous pressure, we have merely to suppose it so constituted that two of the wave-disturbances shall be *accurately* in the wave's front, agreeably to Fresnel's fundamental hypothesis, thence to deduce his general construction for the propagation of waves in biaxal crystals. In fact, we shall afterwards prove that the function ϕ_2, which in its most general form contains twenty-one coefficients, is, in consequence of this hypothesis, reduced to one containing only seven coefficients; and that, from this last form of our function, we obtain for the directions of the disturbance and velocities of propagation precisely the same values as given by Fresnel's construction.

The above supposes, that in a state of equilibrium every part of the medium is quite free from pressure. When this is not the case, A, B, and C will no longer vanish in the equation (8). In the first place, conceive the plane of the wave's front parallel to the plane (yz); then $a = 1$, $b = 0$, $c = 0$, and the equation (8) of our ellipsoid becomes

$$1 = \mu x^2 + A(x^2 + y^2 + z^2) + Mz^2 + Ny^2;$$

and that of a section by a plane through its centre parallel to the wave's front, will be

$$1 = (A + N)\, y^2 + (A + M)\, z^2:$$

and hence, by what precedes, the velocities of propagation of our two polarized waves will be

$\sqrt{A+N}$. The disturbance being parallel to the axis of y,

$\sqrt{A+M}$. .... to the axis of z.

Similarly, if the plane of the wave's front is parallel to the plane (xz), the wave-velocities are

$\sqrt{B+N}$. The disturbance being parallel to the axis x,

$\sqrt{B+L}$. .... to the axis z.

Or if the plane of the wave's front is parallel to (xy), the velocities are

$\sqrt{C+M}$. The disturbance being parallel to x,

$\sqrt{C+L}$. .... y.

Fresnel supposes that the wave-velocity depends on the direction of the disturbance only, and is independent of the position of the wave's front. Instead of assuming this to be generally true, let us merely suppose it holds good for these three principal waves. Then we shall have

$$N+A=C+L, \quad M+A=B+L, \text{ and } B+N=C+M;$$

or we may write

$$A-L=B-M=C-N=\nu \text{ (suppose)}.$$

Thus our equation (8) becomes since $a^2+b^2+c^2=1$,

$$\begin{aligned} 1 &= \mu\,(ax+by+cz)^2+\nu\,(x^2+y^2+z^2) \\ &+ (La^2+Mb^2+Nc^2)\,(x^2+y^2+z^2) \\ &+ L\,(cy-bz)^2+M\,(az-cx)^2+N\,(bx-ay)^2. \end{aligned}$$

But the two last lines of this formula easily reduce to

$$\begin{aligned} &(M+N)\,x^2+(N+L)\,y^2+(L+M)\,z^2 \\ &+ L\,\{a^2x^2-(by+cz)^2\}+M\,\{b^2y^2-(ax+cz)^2\} \\ &+ N\,\{c^2z^2-(ax+by)^2\}. \end{aligned}$$

And hence our last equation becomes

$$\begin{aligned} 1 &= (\nu+M+N)\,x^2+(\nu+N+L)\,y^2+(\nu+L+M)\,z^2+\mu\,(ax+by+cz)^2 \\ &+ L\{a^2x^2-(by+cz)^2\}+M\{b^2y^2-(ax+cz)^2\}+N\{c^2z^2-(ax+by)^2\} \end{aligned}$$
....................(11).

In consequence of the condition which was satisfid in forming the equation (8), it is evident that two of its semi-axes are in a plane parallel to the wave's front, and of which the equation is

$$0 = ax + by + cz \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12);$$

the same therefore will be true for the ellipsoid whose equation is (11), as this is only a particular case of the former. But the section of the last ellipsoid by the plane (12) is evidently given by

$$\left.\begin{array}{l} 1 = (\nu + M + N)\,x^2 + (\nu + L + N)\,y^2 + (\nu + L + M)\,z^2 \\ 0 = ax + by + cz \end{array}\right\} \ldots (12, 1)$$

By what precedes, the two axes of this elliptical section will give the two directions of disturbance which will cause a wave to be propagated without subdivision, and the velocity of propagation of each wave will be inversely as the corresponding semi-axes of the section: which agrees with Fresnel's construction, supposing, as he has done, the actual direction of the disturbance of the particles of the ether is perpendicular to the plane of polarization.

Let us again consider the system as quite free from extraneous pressure and take the most general value of ϕ_2 containing twenty-one coefficients. Then, if to abridge, we make

$$\frac{du}{dx} = \xi, \qquad \frac{du}{dy} = \eta, \qquad \frac{du}{dz} = \zeta;$$

$$\frac{dv}{dz} + \frac{dw}{dy} = \alpha, \quad \frac{du}{dz} + \frac{dw}{dx} = \beta, \quad \frac{du}{dy} + \frac{dv}{dx} = \gamma,$$

we shall have

$$\begin{aligned} -\phi_2 = {} & (\xi^2)\,\xi^2 + (\eta^2)\,\eta^2 + (\zeta^2)\,\zeta^2 + 2\,(\eta\zeta)\,\eta\zeta + 2\,(\xi\zeta)\,\xi\zeta + 2\,(\xi\eta)\,\xi\eta \\ & + (\alpha^2)\,\alpha^2 + (\beta^2)\,\beta^2 + (\gamma^2)\,\gamma^2 + 2\,(\beta\gamma)\,\beta\gamma + 2\,(\alpha\gamma)\,\alpha\gamma + 2\,(\alpha\beta)\,\alpha\beta \\ & + 2\,(\alpha\xi)\,\alpha\xi + 2\,(\beta\xi)\,\beta\xi + 2\,(\gamma\xi)\,\gamma\xi \\ & + 2\,(\alpha\eta)\,\alpha\eta + 2\,(\beta\eta)\,\beta\eta + 2\,(\gamma\eta)\,\gamma\eta \\ & + 2\,(\alpha\zeta)\,\alpha\zeta + 2\,(\beta\zeta)\,\beta\zeta + 2\,(\gamma\zeta)\,\gamma\zeta, \end{aligned}$$

where (ξ^2), (α^2), &c. are the twenty-one coefficients which enter into ϕ_2. Suppose now the equation to the front of a wave is

$$0 = ax + by + cz$$

Then, by what was before observed, the right side of the equation of the ellipsoid, which gives the directions of disturbance of the three polarized waves and their respective velocities, will be had from ϕ, by changing u, v, and w into x, y, and z; also

$$\frac{d}{dx}, \ \frac{d}{dy}, \text{ and } \frac{d}{dz} \text{ into } a,\ b, \text{ and } c.$$

We shall thus get

$$1 = Ax^2 + By^2 + Cz^2 + 2Dyz + 2Exz + 2Fxy.$$

Provided

$$A = (\xi^2)\,a^2 + (\beta^2)\,c^2 + (\gamma^2)\,b^2 + 2\,(\beta\gamma)\,bc + 2\,(\xi\beta)\,ac + 2\,(\xi\gamma)\,ab,$$

$$B = (\eta^2)\,b^2 + (\alpha^2)\,c^2 + (\gamma^2)\,a^2 + 2\,(\alpha\gamma)\,ac + 2\,(\eta\alpha)\,bc + 2\,(\eta\gamma)\,ab,$$

$$C = (\zeta^2)\,c^2 + (\alpha^2)\,b^2 + (\beta^2)\,a^2 + 2\,(\alpha\beta)\,ab + 2\,(\zeta\alpha)\,bc + 2\,(\zeta\beta)\,ac,$$

$$D = (\eta\zeta)\,bc + (\alpha^2)\,bc + (\beta\gamma)\,a^2 + (\alpha\beta)\,ac + (\alpha\gamma)\,ab$$
$$+ (\alpha\eta)\,b^2 + (\alpha\zeta)\,c^2 + (\beta\eta)\,ab + (\gamma\zeta)\,ac,$$

$$E = (\xi\zeta)\,ac + (\beta^2)\,ac + (\alpha\gamma)\,b^2 + (\alpha\beta)\,bc + (\beta\gamma)\,ab$$
$$+ (\beta\xi)\,a^2 + (\beta\zeta)\,c^2 + (\alpha\xi)\,ab + (\gamma\zeta)\,bc,$$

$$F = (\xi\eta)\,ab + (\gamma^2)\,ab + (\alpha\beta)\,c^2 + (\alpha\gamma)\,bc + (\beta\gamma)\,ac$$
$$+ (\gamma\xi)\,a^2 + (\gamma\eta)\,b^2 + (\alpha\xi)\,ac + (\beta\eta)\,bc.$$

But if the directions of two of the disturbances are rigorously in the front of a wave, a plane parallel to this front passing through the centre of the ellipsoid, and whose equation is

$$0 = ax + by + cz,$$

must contain two of the semi-axes of this ellipsoid: and therefore a system of chords perpendicular to the plane will be bisected by it, and hence we get

$$0 = (A - C)\,ac + E\,(c^2 - a^2) + Fbc - Dab,$$

$$0 = (B - C)\,bc + D\,(c^2 - b^2) + Fac - Eab.$$

Substituting in these the values of A, B, &c., before given, we shall obtain the fourteen relations following between the coefficients of ϕ_2, viz.

$$0=(\alpha\eta),\quad 0=(\beta\xi),\quad 0=(\gamma\xi),\quad 0=(\alpha\zeta),\quad 0=(\beta\zeta),\quad 0=(\gamma\eta),$$

$$(\alpha\xi)=-2(\beta\gamma),\quad (\beta\eta)=-2(\alpha\gamma),\quad (\gamma\zeta)=-2(\alpha\beta),$$

$$(\xi^2)=(\eta^2)=(\zeta^2)=2(\alpha^2)+(\eta\zeta)=2(\beta^2)+(\xi\zeta)=2(\gamma^2)+(\xi\eta).$$

Hence, we may readily put the function ϕ_2 under the following form,

$$(\xi^2)(\xi+\eta+\zeta)^2+(\alpha^2)(\alpha^2-4\eta\zeta)+(\beta^2)(\beta^2-4\xi\zeta)+(\gamma^2)(\gamma^2-4\xi\eta)$$
$$+2(\beta\gamma)(\beta\gamma-2\alpha\xi)+2(\alpha\gamma)(\alpha\gamma-2\beta\eta)+2(\alpha\beta)(\alpha\beta-2\gamma\zeta),$$

or by restoring the values of ξ, η, &c., and making $G=(\xi^2)$, $L=(\alpha^2)$, &c., our function will become

$$G\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)^2+$$
$$L\left\{\left(\frac{dv}{dz}+\frac{dw}{dy}\right)^2-4\frac{dv}{dy}\frac{dw}{dz}\right\}+M\left\{\left(\frac{du}{dz}+\frac{dw}{dx}\right)^2-4\frac{du}{dx}\frac{dw}{dz}\right\}+N\left\{\left(\frac{du}{dy}+\frac{dv}{dz}\right)^2-4\frac{du}{dx}\frac{dv}{dy}\right\}$$
$$+2P\left\{\left(\frac{du}{dz}+\frac{dw}{dx}\right)\left(\frac{du}{dy}+\frac{dv}{dx}\right)-2\frac{du}{dx}\left(\frac{dv}{dz}+\frac{dw}{dy}\right)\right\}$$
$$+2Q\left\{\left(\frac{dv}{dz}+\frac{dw}{dy}\right)\left(\frac{du}{dy}+\frac{dv}{dx}\right)-2\frac{dv}{dy}\left(\frac{du}{dz}+\frac{dw}{dx}\right)\right\}$$
$$+2R\left\{\left(\frac{dv}{dz}+\frac{dw}{dy}\right)\left(\frac{du}{dz}+\frac{dw}{dx}\right)-2\frac{dw}{dz}\left(\frac{du}{dy}+\frac{dv}{dx}\right)\right\}\ \ldots\ldots(12),$$

and hence we get for the equation of the corresponding ellipsoid,

$$1=G(ax+by+cz)^2+L(bz-cy)^2+M(az-cx)^2+N(ay-bx)^2$$
$$+2P(cx-az)(ay-bx)+2Q(bz-cy)(ay-bx)+2R(bz-cy)(cx-az)\ldots(13)$$

But if in equation (8) and corresponding function (A), we suppose $A=0$, $B=0$, and $C=0$, and then refer the equation to axes taken arbitrarily in space, we shall thus introduce three new coefficients, and evidently obtain a result equivalent to equation (13) and function (12). We therefore see that the single supposition of the wave-disturbance, being always *accurately* in the wave's front, leads to a result equivalent to that given by the former process; and we are thus assured that by employing the simpler method we do not, in the case in question, eventually lessen the generality of our result, but merely, in effect, select the three rectangular axes, which may be called the axes of elasticity of the medium, for our co-ordinate axes. From the general form of ϕ, it is clear that the same observation applies to it, and therefore the consequences before deduced possess all the requisite generality.

The same conclusions may be obtained, whether we introduce the consideration of extraneous pressures or not, by direct calculation. In fact, when these pressures vanish, and we conceive a section of the ellipsoid whose equation is (13), made by a plane parallel to the wave's front, to turn 90 degrees in its own plane the same reasoning by which equation (10) was before found, immediately gives, in the present case,

$$1 = Lx'^2 + My'^2 + Nz'^2 + 2Py'z' + 2Qx'z' + 2Rx'y' \ldots (14),$$

for the equation of the surface in which all the elliptical sections in their new situations, and corresponding to every position of the wave's front, will be found.

Lastly, when we introduce the consideration of extraneous pressures, it is clear, from what precedes, that we shall merely have to add to the function on the right side of the equation (13), the quantity

$$(Aa^2 + Bb^2 + Cc^2 + 2Dbc + 2Fac + 2Fab)(x^2 + y^2 + z^2),$$

which would arise from changing u, v, and w into x, y, and z. Also $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$ into a, b, c; in that part of ϕ which is of the second degree in u, v, w, agreeably to the remark in a foregoing note. Afterwards, when we determine the values of A, B, &c., by the same condition which enabled us to deduce the system (12, 1), we shall have, in the place of this system, the following:

$$\left.\begin{array}{l} 1=K(x^2+y^2+z^2)-\{Lx^2+My^2+Nz^2+2Pyz+2Qxz+2Rxy\} \\ 0=ax+by+cz \end{array}\right\} \ldots 15),$$

which is applicable to the more general case just considered.*

* Vide Professor Stokes' Report on Double Refraction (British Association, 1862, p. 265).