ON THE LAWS

OF

REFLEXION AND REFRACTION OF LIGHT

AT THE COMMON SURFACE OF TWO NON-CRYSTALLIZED MEDIA.

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ON THE LAWS OF THE REFLEXION AND REFRACTION OF LIGHT AT THE COMMON SURFACE OF TWO NON-CRYSTALLIZED MEDIA.

M. Cauchy seems to have been the first who saw fully the utility of applying to the Theory of Light those formulæ which represent the motions of a system of molecules acting on each other by mutually attractive and repulsive forces; supposing always that in the mutual action of any two particles, the particles may be regarded as points animated by forces directed along the right line which joins them. This last supposition, if applied to those compound particles, at least, which are separable by mechanical division, seems rather restrictive; as many phenomena, those of crystallization for instance, seem to indicate certain polarities in these particles. If, however, this were not the case, we are so perfectly ignorant of the mode of action of the elements of the luminiferous ether on each other, that it would seem a safer method to take some general physical principle as the basis of our reasoning, rather than assume certain modes of action, which, after all, may be widely different from the mechanism employed by nature; more especially if this principle include in itself, as a particular case, those before used by M. Cauchy and others, and also lead to a much more simple process of calculation. The principle selected as the basis of the reasoning contained in the following paper is this: In whatever way the elements of any material system may act upon each other, if all the internal forces exerted be multiplied by the elements of their respective directions, the total sum for any assigned portion of the mass will always be the exact differential of some function. But, this function being known, we can immediately apply the general method given in the *Mécanique Analytique*, and which appears to be more especially applicable to problems that relate to the motions of systems composed of an immense number of particles mutually acting upon each other. One of the advantages of this method, of great importance, is, that we are necessarily led by the mere process of the calculation, and with little care on our part, to all the equations and conditions which are *requisite* and *sufficient* for the complete solution of any problem to which it may be applied.

The present communication is confined almost entirely to the consideration of non-crystallized media; for which it is proved, that the function due to the molecular actions, in its most general form, contains only two arbitrary coefficients, A and B; the values of which depend of course on the unknown internal constitution of the medium under consideration, and it would be easy to shew, for the most general case, that any arbitrary disturbance, excited in a very small portion of the medium, would in general give rise to two spherical waves, one propagated entirely by normal, the other entirely by transverse, vibrations, and such that if the velocity of transmission of the former wave be represented by $\sqrt{A}$, that of the latter would be represented by $\sqrt{B}$. But in the transmission of light through a prism, though the wave which is propagated by normal vibrations were incapable itself of affecting the eye, yet it would be capable of giving rise to an ordinary wave of light propagated by transverse vibrations, except in the extreme cases where $\frac{A}{B} = 0$, or $\frac{A}{B} =$ a very large quantity; which, for the sake of simplicity, may be regarded as infinite; and it is not difficult to prove that the equilibrium of our medium would be unstable unless $\frac{A}{B} > \frac{4}{3}$. We are therefore compelled to adopt the latter value of $\frac{A}{B}$, and thus to admit that in the luminiferous ether, the velocity of transmission of waves propagated by normal vibrations is very great compared with that of ordinary light.

The principal results obtained in this paper relate to the tensity of the wave reflected at the common surface of two media, both for light polarized in and perpendicular to the plane of incidence; and likewise to the change of phase which takes place when the reflexion becomes total. In the former case, our values agree precisely with those given by Fresnel; supposing, as he has done, that the direction of the actual motion of the particles of the luminiferous ether is perpendicular to the plane of polarization. But it results from our formulæ, when the light is polarized perpendicular to the plane of incidence, that the expressions given by Fresnel are only very near approximations; and that the intensity of the reflected wave will never become absolutely null, but only attain a minimum value; which, in the case of reflexion from water at the proper angle, is $\frac{1}{181}$ part of that of the incident wave. This minimum value increases rapidly, as the index of refraction increases, and thus the quantity of light reflected at the polarizing angle, becomes considerable for highly refracting substances, a fact which has been long known to experimental philosophers.

It may be proper to observe, that M. Cauchy (*Bulletin des Sciences*, 1830) has given a method of determining the intensity of the waves reflected at the common surface of two media. He has since stated, (*Nouveaux Exercises des Mathématiques,*) that the hypothesis employed on that occasion is inadmissible, and has promised in a future memoir, to give a *new mechanical principle* applicable to this and other questions; but I have not been able to learn whether such a memoir has yet appeared. The first method consisted in satisfying a part, and only a part, of the conditions belonging to the surface of junction, and the consideration of the waves propagated by normal vibrations was wholly overlooked, though it is easy to perceive, that in general waves of this kind must necessarily be produced when the incident wave is polarized perpendicular to the plane of incidence, in consequence of the incident and refracted waves being in different planes. Indeed, without introducing the consideration of these last waves, it is impossible to satisfy the whole of the conditions due to the surface of junction of the two media. But when this consideration is introduced, the whole of the conditions may be satisfied, and the principles given in the *Mécanique Analytique* became abundantly sufficient for the solution of the problem.

In conclusion, it may be observed, that the radius of the sphere of sensible action of the molecular forces has been regarded as insensible with respect to the length λ of a wave of light, and thus, for the sake of simplicity, certain terms have been disregarded on which the different refrangibility of differently coloured rays might be supposed to depend. These terms, which are necessary to be considered when we are treating of the dispersion, serve only to render our formulæ uselessly complex in other investigations respecting the phenomena of light.

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Let us conceive a mass composed of an immense number of molecules acting on each other by any kind of molecular forces, but which are sensible only at insensible distances, and let moreover the whole system be quite free from all extraneous action of every kind. Then x, y and z being the co-ordinates of any particle of the medium under consideration when in equilibrium, and

$$x + u, \quad y + v, \quad z + w,$$

the co-ordinates of the same particle in a state of motion (where u, v, and w are very small functions of the original co-ordinates (x, y, z), of any particle and of the time (t)), we get, by combining D'Alembert's principle with that of virtual velocities,

$$\Sigma Dm \left\{\frac{d^2u}{dt^2}\delta u + \frac{d^2v}{dt^2}\delta v + \frac{d^2w}{dt^2}\delta w\right\} = \Sigma Dv \,.\, \delta\phi \,\ldots\ldots.. (1);$$

Dm and Dv being exceedingly small corresponding elements of the mass and volume of the medium, but which nevertheless contain a very great number of molecules, and $\delta\phi$ the exact differential of some function and entirely due to the internal actions of the particles of the medium on each other. Indeed, if $\delta\phi$ were not an exact differential, a perpetual motion would be possible, and we have every reason to think, that the forces in nature are so disposed as to render this a natural impossibility.

Let us now take any element of the medium, rectangular in a state of repose, and of which the sides are dx, dy, dz; the length of the sides composed of the same particles will in a state of motion become

$$dx' = dx(1+s_1), \quad dy' = dy(1+s_2), \quad dz' = dz(1+s_3);$$

where s_1, s_2, s_3 are exceedingly small quantities of the first order. If, moreover, we make,

$$\alpha = \cos < \frac{dy'}{dz'}, \quad \beta = \cos < \frac{dx'}{dz'}, \quad \gamma = \cos < \frac{dx'}{dy};$$

α, β, and γ will be very small quantities of the same order. But, whatever may be the nature of the internal actions, if we represent by

$$\delta\phi\, dx\, dy\, dz,$$

the part of the second member of the equation (1), due to the molecules in the element under consideration, it is evident, that ϕ will remain the same when all the sides and all the angles of the parallelopiped, whose sides are dx' dy' dz', remain unaltered, and therefore its most general value must be of the form

$$\phi = \text{function } \{s_1, s_2, s_3, \alpha, \beta, \gamma\}.$$

But s_1, s_2, s_3, α, β, γ being very small quantities of the first order, we may expand ϕ in a very convergent series of the form

$$\phi = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \&c.:$$

ϕ_0, ϕ_1, ϕ_2, &c. being homogeneous functions of the six quantities $\alpha, \beta, \gamma, s_1, s_2, s_3$ of the degrees 0, 1, 2, &c. each of which is very great compared with the next following one. If now, ρ represent the primitive density of the element $dx\, dy\, dz$, we may write $\rho\, dx\, dy\, dz$ in the place of Dm in the formula (1), which will thus become, since ϕ_0 is constant

$$\iiint \rho\, dx\, dy\, dz \left\{ \frac{d^2u}{dt^2}\, \delta u + \frac{d^2v}{dt^2}\, \delta v + \frac{d^2w}{dt^2}\, \delta w \right\}$$

$$= \iiint dx\, dy\, dz\, (\delta\phi_1 + \delta\phi_2 + \&c.)\,;$$

the triple integrals extending over the whole volume of the medium under consideration.

But by the supposition, when $u = 0$, $v = 0$ and $w = 0$, the system is in equilibrium, and hence

$$0 = \iiint dx\, dy\, dz\, \delta\phi_1 :$$

seeing that ϕ_1 is a homogeneous function of s_1, s_2, s_3, α, β, γ of the *first* degree only. If therefore we neglect ϕ_3, ϕ_4, &c. which are exceedingly small compared with ϕ_2, our equation becomes

$$\iiint \rho\, dx\, dy\, dz \left\{ \frac{d^2u}{dt^2}\, \delta u + \frac{d^2v}{dt^2}\, \delta v + \frac{d^2w}{dt^2}\, \delta w \right\} = \iiint dx\, dy\, dz\, \delta\phi_2 \ldots (2)\,;$$

the integrals extending over the whole volume under consideration. The formula just found is true for any number of media comprised in this volume, provided the whole system be perfectly free from all extraneous forces, and subject only to its own molecular actions.

If now we can obtain the value of ϕ_2, we shall only have to apply the general methods given in the *Mécanique Analytique.* But ϕ_2 being a homogeneous function of six quantities of the second degree, will in its most general form contain 21 arbitrary coefficients. The proper value to be assigned to each will of course depend on the internal constitution of the medium. If, however, the medium be a non-crystallized one, the form of ϕ_2 will remain the same, whatever be the directions of the co-ordinate axes in space. Applying this last consideration, we shall find that the most general form of ϕ_2 for non-crystallized bodies contains only two arbitrary coefficients. In fact, by neglecting quantities of the higher orders, it is easy to perceive that

$$s_1 = \frac{du}{dx}, \quad s_2 = \frac{dv}{dy}, \quad s_3 = \frac{dw}{dz},$$

$$\alpha = \frac{dw}{dy} + \frac{dv}{dz}, \quad \beta = \frac{dw}{dx} + \frac{du}{dz}, \quad \gamma = \frac{du}{dy} + \frac{dv}{dx},$$

and if the medium is symmetrical with regard to the plane (xy) only, ϕ_2 will remain unchanged when $-z$ and $-w$ are written for z and w. But this alteration evidently changes α and β to $-\alpha$ and $-\beta$. Similar observations apply to the planes (xz) (yz). If therefore the medium is merely symmetrical with respect to each of the three co-ordinate planes, we see that ϕ_2 must remain unaltered when

$$\left.\begin{array}{l} \text{or } -z,\ -w,\ -\alpha,\ -\beta \\ \text{or } -y,\ -v,\ -\alpha,\ -\gamma \\ \text{or } -x,\ -u,\ -\beta,\ -\gamma \end{array}\right\} \text{are written for} \left\{\begin{array}{l} z,\ w,\ \alpha,\ \beta \\ y,\ v,\ \alpha,\ \gamma \\ x,\ u,\ \beta,\ \gamma. \end{array}\right.$$

In this way the 21 coefficients are reduced to 9, and the resulting function is of the form

$$G\left(\frac{du}{dx}\right)^2 + H\left(\frac{dv}{dy}\right)^2 + I\left(\frac{dw}{dz}\right)^2 + L\alpha^2 + M\beta^2 + N\gamma^2$$

$$+ 2P\frac{dv}{dy}\cdot\frac{dw}{dz} + 2Q\frac{du}{dx}\cdot\frac{dw}{dz} + 2R\frac{du}{dx}\cdot\frac{dv}{dy} = \phi_2 \ldots (A).$$

Probably the function just obtained may belong to those crystals which have three axes of elasticity at right angles to each other.

Suppose now we further restrict the generality of our function by making it symmetrical all round one axis, as that of z for instance. By shifting the axis of x through the infinitely small angle $\delta\theta$,

$$\left.\begin{array}{c} x \\ y \\ z \end{array}\right\} \text{ becomes } \left\{\begin{array}{l} x + y\delta\theta \\ y - x\delta\theta, \\ z \end{array}\right.$$

$$\left.\begin{array}{c}\dfrac{d}{dx}\\[2ex]\dfrac{d}{dy}\\[2ex]\dfrac{d}{dz}\end{array}\right\}\text{ becomes }\left\{\begin{array}{l}\dfrac{d}{dx}+\delta\theta\,\dfrac{d}{dy}\\[2ex]\dfrac{d}{dy}-\delta\theta\,\dfrac{d}{dx},\\[2ex]\dfrac{d}{dz}\end{array}\right.$$

and

$$\left.\begin{array}{c}u\\ v\\ w\end{array}\right\}\text{ becomes }\left\{\begin{array}{l}u+v\delta\theta\\ v-u\delta\theta.\\ w\end{array}\right.$$

Making these substitutions in (A), we see that the form of ϕ_2 will not remain the same for the new axes, unless

$$G=H=2N+R,$$
$$L=M,$$
$$P=Q;$$

and thus we get

$$\phi_2=G\left\{\left(\frac{du}{dx}\right)^2+\left(\frac{dv}{dy}\right)^2\right\}+I\left(\frac{dw}{dz}\right)^2+L\left(\alpha^2+\beta^2\right)$$
$$+N\gamma^2+2P\left(\frac{dv}{dy}+\frac{du}{dx}\right)\frac{dw}{dz}+(2G-4N)\frac{du}{dx}\cdot\frac{dv}{dy}\ldots(B);$$

under which form it may possibly be applied to uniaxal crystals.

Lastly, if we suppose the function ϕ_2 symmetrical with respect to all three axes, there results

$$G=H=I=2N+R,$$
$$L=M=N,$$
$$P=Q=R;$$

and consequently,

$$\phi^2=G\left\{\left(\frac{du}{dx}\right)^2+\left(\frac{dv}{dy}\right)^2+\left(\frac{dw}{dz}\right)^2\right\}+L\left(\alpha^2+\beta^2+\gamma^2\right)$$
$$+(2G-4L)\left\{\frac{dv}{dy}\cdot\frac{dw}{dz}+\frac{du}{dx}\cdot\frac{dw}{dz}+\frac{du}{dx}\cdot\frac{dv}{dy}\right\};$$

or, by merely changing the two constants and restoring the values of α, β, and γ,

$$2\phi_2 = -A\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)^2$$

$$-B\left\{\left(\frac{du}{dy}+\frac{dv}{dx}\right)^2+\left(\frac{du}{dz}+\frac{dw}{dx}\right)^2+\left(\frac{dv}{dz}+\frac{dw}{dy}\right)^2\right.$$

$$\left.-4\left(\frac{dv}{dy}\cdot\frac{dw}{dz}+\frac{du}{dx}\cdot\frac{dw}{dz}+\frac{du}{dx}\cdot\frac{dv}{dy}\right)\right\}\ \ldots\ldots\ (C).$$

This is the most general form that ϕ_2 can take for non-crystallized bodies, in which it is perfectly indifferent in what directions the rectangular axes are placed. The same result might be obtained from the most general value of ϕ_2, by the method before used to make ϕ_2 symmetrical all round the axis of z, applied also to the other two axes. It was, indeed, thus I first obtained it. The method given in the text, however, and which is very similar to one used by M. Cauchy, is not only more simple, but has the advantage of furnishing two intermediate results, which may possibly be of use on some future occasion.

Let us now consider the particular case of two indefinitely extended media, the surface of junction when in equilibrium being a plane of infinite extent, horizontal (suppose), and which we shall take as that of (yz), and conceive the axis of x positive directed downwards. Then if ρ be the constant density of the upper, and $\rho_{,}$ that of the lower medium, ϕ_2 and $\phi_2^{(1)}$ the corresponding functions due to the molecular actions; the equation (2) adapted to the present case will become

$$\iiint \rho\, dx\, dy\, dz \left\{\frac{d^2u}{dt^2}\,\delta u+\frac{d^2v}{dt^2}\,\delta v+\frac{d^2w}{dt^2}\,\delta w\right\}$$

$$+\iiint \rho_{,}\, dx\, dy\, dz \left\{\frac{d^2u_{,}}{dt^2}\,\delta u_{,}+\frac{d^2v_{,}}{dt^2}\,\delta v_{,}+\frac{d^2w_{,}}{dt^2}\,dw_{,}\right\},$$

$$=\iiint dx\, dy\, dz\, \phi_2+\iiint dx\, dy\, dz\, \phi_2^{(1)}\ \ldots\ldots\ldots\ (3);$$

$u_{,}$, $v_{,}$, $w_{,}$ belonging to the lower fluid, and the triple integrals being extended over the whole volume of the fluids to which they respectively belong.

It now only remains to substitute for ϕ_1 and $\phi_1^{(1)}$ their values, to effect the integrations by parts, and to equate separately to zero the coefficients of the independent variations. Substituting therefore for ϕ_1 its value (C), we get

$$\iiint dx\,dy\,dz\,\delta\phi_1$$

$$= -A\iiint dx\,dy\,dz\left\{\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\left(\frac{d\delta u}{dx}+\frac{d\delta v}{dy}+\frac{d\delta w}{dz}\right)\right\}$$

$$-B\iiint dx\,dy\,dz\left\{\left(\frac{du}{dy}+\frac{dv}{dx}\right)\left(\frac{d\delta u}{dy}+\frac{d\delta v}{dx}\right)+\left(\frac{du}{dz}+\frac{dw}{dx}\right)\left(\frac{d\delta u}{dz}+\frac{d\delta w}{dx}\right)+\left(\frac{dv}{dz}+\frac{dw}{dy}\right)\left(\frac{d\delta v}{dz}+\frac{d\delta w}{dy}\right)\right.$$

$$\left.-2\left[\left(\frac{dv}{dy}\cdot\frac{d\delta w}{dz}+\frac{dw}{dz}\cdot\frac{d\delta v}{dy}\right)+\left(\frac{du}{dx}\cdot\frac{d\delta w}{dz}+\frac{dw}{dz}\cdot\frac{d\delta u}{dx}\right)+\left(\frac{du}{dx}\cdot\frac{d\delta v}{dy}+\frac{dv}{dy}\cdot\frac{d\delta u}{dx}\right)\right]\right\}$$

$$= -\iint dy\,dz\left\{A\cdot\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)-2B\left(\frac{dv}{dy}+\frac{dw}{dz}\right)\right\}.\delta u$$

$$-\iint dy\,dz\left\{B\left(\frac{du}{dy}+\frac{dv}{dx}\right)\delta v+B\left(\frac{du}{dz}+\frac{dw}{dx}\right)\delta w\right\}$$

$$+\iiint dx\,dy\,dz\left\{A\frac{d}{dx}\cdot\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)+B\left[\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}-\frac{d}{dx}\left(\frac{dv}{dy}+\frac{dw}{dz}\right)\right]\right\}.\delta u$$

$$+\left\{A\frac{d}{dy}\cdot\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)+B\left[\frac{d^2v}{dx^2}+\frac{d^2v}{dz^2}-\frac{d}{dy}\left(\frac{du}{dx}+\frac{dw}{dz}\right)\right]\right\}\delta v$$

$$+\left\{A\frac{d}{dz}\cdot\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)+B\left[\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}-\frac{d}{dz}\cdot\left(\frac{du}{dx}+\frac{dv}{dy}\right)\right]\right\}\delta w;$$

seeing that we may neglect the double integrals at the limits $x=-\infty$, $y=\pm\infty$, $z=\pm\infty$; as the conditions imposed at these limits cannot affect the motion of the system at any *finite* distance from the origin; and thus the double integrals belong only to the surface of junction, of which the equation, in a state of equilibrium, is

$$0=x.$$

In like manner we get

$$\iiint dx\,dy\,dz\,\delta\phi_{,}^{(1)}$$

$$= + \iint dy\,dz \left\{ A_{,} \left(\frac{du_{,}}{dx} + \frac{dv_{,}}{dy} + \frac{dw_{,}}{dz}\right) - 2B_{,} \left(\frac{dv_{,}}{dy} + \frac{dw_{,}}{dz}\right)\right\} \delta u_{,}$$

$$+ \iint dy\,dz \left\{ B \left(\frac{du_{,}}{dy} + \frac{dv_{,}}{dx}\right) \delta v_{,} + B \left(\frac{du_{,}}{dz} + \frac{dw_{,}}{dx}\right) \delta w_{,} \right\}$$

+ the triple integral;

since it is the *least* value of x which belongs to the surface of junction in the *lower* medium, and therefore the double integrals belonging to the limiting surface must have their signs changed.

If, now, we substitute the preceding expression in (3), equate separately to zero the coefficients of the independent variation δu, δv, δw, under the triple sign of integration, there results for the upper medium

$$\rho \frac{d^2 u}{dt^2} = A \frac{d}{dx} \cdot \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) + B \left\{\frac{d^2 u}{dy^2} + \frac{d^2 u}{dz^2} - \frac{d}{dx} \cdot \left(\frac{dv}{dy} + \frac{dw}{dz}\right)\right\};$$

$$\rho \frac{d^2 v}{dt^2} = A \frac{d}{dy} \cdot \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) + B \left\{\frac{d^2 v}{dx^2} + \frac{d^2 v}{dz^2} - \frac{d}{dy} \cdot \left(\frac{du}{dx} + \frac{dw}{dz}\right)\right\} (4);$$

$$\rho \frac{d^2 w}{dt^2} = A \frac{d}{dz} \cdot \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) + B \left\{\frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} - \frac{d}{dz} \cdot \left(\frac{du}{dx} + \frac{dv}{dy}\right)\right\};$$

and by equating the coefficients of $\delta u_{,}$, $\delta v_{,}$, $\delta w_{,}$, we get three similar equations for the lower medium.

To the six general equations just obtained, we must add the conditions due to the surface of junction of the two media; and at this surface we have first,

$$u = u_{,}, \quad v = v_{,}, \quad w = w_{,}, \quad (\text{when } x = 0) \ldots\ldots\ldots (5);$$

and consequently,

$$\delta u = \delta u_{,}; \quad \delta v = \delta v_{,}; \quad \delta w = \delta w_{,}.$$

But the part of the equation (3) belonging to this surface, and which yet remains to be satisfied, is

$$0 = -\iint dydz \left\{A\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) - 2B\left(\frac{dv}{dy} + \frac{dw}{dz}\right)\right\} \delta u$$

$$+ \iint dydz \left\{A_{,}\left(\frac{du_{,}}{dx} + \frac{dv_{,}}{dy} + \frac{dw_{,}}{dz}\right) - 2B_{,}\left(\frac{dv_{,}}{dy} + \frac{dw_{,}}{dz}\right)\right\} \delta u_{,}$$

$$+ \iint dydz \left\{B\left(\frac{du}{dy} + \frac{dv}{dx}\right)\delta v + B\left(\frac{du}{dz} + \frac{dw}{dx}\right)\delta w\right\}$$

$$+ \iint dydz \left\{B_{,}\left(\frac{du_{,}}{dy} + \frac{dv_{,}}{dx}\right)\delta v_{,} + B_{,}\left(\frac{du_{,}}{dz} + \frac{dw_{,}}{dx}\right)\delta w_{,}\right\};$$

and as $\delta u = \delta u_{,}$, &c., we obtain, as before,

$$A\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) - 2B\left(\frac{dv}{dy} + \frac{dw}{dz}\right)$$

$$= A_{,}\left(\frac{du_{,}}{dx} + \frac{dv_{,}}{dy} + \frac{dw_{,}}{dz}\right) - 2B_{,}\left(\frac{dv_{,}}{dy} + \frac{dw_{,}}{dz}\right)$$

$$B\left(\frac{du}{dy} + \frac{dv}{dx}\right) = B_{,}\left(\frac{du_{,}}{dy} + \frac{dv_{,}}{dx}\right) \quad \ldots\ldots\ldots\ldots\ldots (6),$$

$$B\left(\frac{du}{dz} + \frac{dw}{dx}\right) = B_{,}\left(\frac{du_{,}}{dz} + \frac{dw_{,}}{dx}\right);$$

and these belong to the particular value $x = 0$.

The six particular conditions (5) and (6), belonging to the surface of junction of the two media, combined with the six general equations before obtained, are *necessary* and *sufficient* for the complete determination of the motion of the two media, supposing the initial state of each given. We shall not here attempt their general solution, but merely consider the propagation of a plane wave of infinite extent, accompanied by its reflected and refracted waves, as in the preceding paper on Sound.

Let the direction of the axis of z, which yet remains arbitrary, be taken parallel to the intersection of the plane of the incident wave with the surface of junction, and suppose the disturbance of the particles to be wholly in the direction of the axis of z, which is the case with light polarized in the plane of incidence, according to Fresnel. Then we have

$$0 = u, \quad 0 = v, \quad 0 = u_{,} \quad 0 = v_{,};$$

and supposing the disturbance the same for every point of the same front of a wave, w and $w_{,}$ will be independent of z, and thus the three general equations (4) will all be satisfied if

$$\rho \frac{d^2 w}{dt^2} = B \left\{ \frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} \right\},$$

or by making $\frac{B}{\rho} = \gamma^2$,

$$\frac{d^2 w}{dt^2} = \gamma^2 \left\{ \frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} \right\} \qquad (7).$$

Similarly in the lower medium we have

$$\frac{d^2 w_{,}}{dt^2} = \gamma_{,}^2 \left\{ \frac{d^2 w_{,}}{dx^2} + \frac{d^2 w_{,}}{dy^2} \right\} \qquad (8),$$

$w_{,}$ and $\gamma_{,}$ belonging to this medium.

It now remains to satisfy the conditions (5) and (6). But these are all satisfied by the preceding values provided

$$w = w_{,},$$

$$B \frac{dw}{dx} = B_{,} \frac{dw_{,}}{dx}.$$

The formulæ which we have obtained are quite general, and will apply to the ordinary elastic fluids by making $B = 0$. But for all the known gases, A is independent of the nature of the gas, and consequently $A = A_{,}$. If, therefore, we suppose $B = B_{,}$ at least when we consider those phenomena only which depend merely on different states of the same medium, as is the case with light, our conditions become*

$$\left.\begin{aligned} w &= w_{,} \\ \frac{dw}{dx} &= \frac{dw_{,}}{dx} \end{aligned}\right\} \text{(when } x = 0) \qquad (9).$$

* Though for all known gases A is independent of the nature of the gas, perhaps it is extending the analogy rather too far, to assume that in the lumini-

The disturbance in the upper medium which contains the incident and reflected wave, will be represented, as in the case of Sound, by

$$w = f(ax + by + ct) + F(-ax + by + ct);$$

f belonging to the incident, F to the reflected plane wave, and c being a negative quantity. Also in the lower medium,

$$w_{,} = f_{,}(a_{,}x + by + ct).$$

These values evidently satisfy the general equation (7) and (8), provided $c^2 = \gamma^2 (a^2 + b^2)$, and $c^2 = \gamma_{,}^2 (a_{,}^2 + b^2)$; we have therefore only to satisfy the conditions (9), which give

$$f(by + ct) + F(by + ct) = f_{,}(by + ct),$$

$$af'(by + ct) - aF'(by + ct) = a_{,}f'(by + ct).$$

Taking now the differential coefficient of the first equation, and writing to abridge the characteristics of the functions only, we get

$$2f' = \left(1 + \frac{a_{,}}{a}\right)f'_{,}, \text{ and } 2F' = \left(1 - \frac{a_{,}}{a}\right)f'_{,},$$

and therefore

$$\frac{F'}{f'} = \frac{1 - \frac{a_{,}}{a}}{1 + \frac{a_{,}}{a}} = \frac{a - a_{,}}{a + a_{,}} = \frac{\cot\theta - \cot\theta_{,}}{\cot\theta + \cot\theta_{,}} = \frac{\sin(\theta_{,} - \theta)}{\sin(\theta_{,} + \theta)};$$

θ and $\theta_{,}$ being the angles of incidence and refraction.

This ratio between the intensity of the incident and reflected

ferous ether the constants A and B must always be independent of the state of the other, as found in different refracting substances. However, since this hypothesis greatly simplifies the equations due to the surface of junction of the two media, and is itself the most simple that could be selected, it seemed natural first to deduce the consequences which follow from it before trying a more complicated one, and, as far as I have yet found, these consequences are in accordance with observed facts.

waves is exactly the same as that for light polarized in the plane of incidence (vide Airy's *Tracts*, p. 356*), and which Fresnel supposes to be propagated by vibrations perpendicular to the plane of incidence, agreeably to what has been assumed in the foregoing process.

We will now limit the generality of the functions f, F and $f_{,}$, by supposing the law of the motion to be similar to that of a cycloidal pendulum; and if we farther suppose the angle of incidence to be increased until the refracted wave ceases to be transmitted in the regular way, as in our former paper on Sound, the proper integral of the equation

$$\frac{d^2 w_{,}}{dt^2} = \gamma_{,}^2 \left\{ \frac{d^2 w_{,}}{dx^2} + \frac{d^2 w_{,}}{dy^2} \right\}$$

will be

$$w_{,} = \epsilon^{-a_{,}' x} B \sin \psi \ldots\ldots\ldots\ldots\ldots\ldots (10);$$

where $\psi = by + ct$, and $a_{,}'$ is determined by

$$\gamma_{,}^2 (b^2 - a_{,}'^2) = c^2 = \gamma^2 (b^2 + a^2) \ldots\ldots\ldots\ldots (11).$$

But one of the conditions (9) will introduce *sines* and the other *cosines*, in such a way that it will be impossible to satisfy them unless we introduce both *sines* and *cosines* into the value of w, or, which amounts to the same, unless we make

$$w = a \sin (ax + by + ct + e) + \beta \sin (-ax + by + ct + e_{,}) \ldots (12),$$

in the first medium, instead of

$$w = a \sin (ax + by + ct) + \beta \sin (-ax + by + ct).$$

which would have been done had the refracted wave been transmitted in the usual way, and consequently no exponential been introduced into the value of $w_{,}$. We thus see the analytical reason for what is called the change of phase which takes place when the reflexion of light becomes total.

* [Airy on the Undulatory Theory of Optics, p. 109, Art. 128.]

Substituting now (10) and (12), in the equations (9), and proceeding precisely as for Sound, we get

$$0 = \alpha \cos e - \beta \cos e_{,},$$

$$0 = \alpha \sin e + \beta \sin e_{,},$$

$$\frac{a_{\iota}'}{a} B = \alpha \sin e - \beta \sin e_{,}$$

$$B = \alpha \cos e + \beta \cos e_{,}.$$

Hence there results $\alpha = \beta$, and $e_{,} = -e$, and

$$\tan e = \frac{a_{\iota}'}{a} = \frac{a_{\iota}'}{b} \div \frac{a}{b} = \frac{a_{\iota}'}{b} \tan \theta.$$

But by (11),

$$\frac{a_{\iota}'}{b} = \sqrt{\left\{1 - \frac{\gamma^2}{\gamma_{\iota}^2} \cdot \left(1 + \frac{a^2}{b^2}\right)\right\}} = \sqrt{\left(1 - \frac{1}{\mu^2 \sin^2 \theta}\right)};$$

by introducing μ the index of refraction, and θ the angle of incidence. Thus,

$$\tan e = \frac{\sqrt{(\mu^2 \sin^2 \theta - 1)}}{\mu \cos \theta};$$

and as e represents half the alteration of phase in passing from the incident to the reflected wave, we see that here also our result agrees precisely with Fresnel's for light polarized in the plane of incidence. (Vide Airy's *Tracts*, p. 362 *.)

Let us now conceive the direction of the transverse vibrations in the incident wave to be perpendicular to the direction in the case just considered; and therefore that the actual motions of the particles are all parallel to the intersection of the plane of incidence (xy) with the front of the wave. Then, as the planes of the incident and refracted waves do not coincide, it is easy to perceive that at the surface of junction there will, in this case, be a resolved part of the disturbance in the direction of the

* [Airy, *ubi sup.* p. 114, Art. 153.]

normal; and therefore, besides the incident wave, there will, in general, be an accompanying reflected and refracted wave, in which the vibrations are transverse, and another pair of accompanying reflected and refracted waves, in which the directions of the vibrations are normal to the fronts of the waves. In fact, unless the consideration of the two latter waves is also introduced, it is impossible to satisfy all the conditions at the surface of junction; and these are as essential to the complete solution of the problem, as the general equations of motion.

The direction of the disturbance being in plane (xy) $w=0$, and as the disturbance of every particle in the same front of a wave is the same, u and v are independent of z. Hence, the general equations (4) for the first medium become

$$\frac{d^2u}{dt^2}=g^2\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}\right)+\gamma^2\frac{d}{dy}\left(\frac{du}{dy}-\frac{dv}{dx}\right),$$

$$\frac{d^2v}{dt^2}=g^2\frac{d}{dy}\left(\frac{du}{dx}+\frac{dv}{dy}\right)+\gamma^2\frac{d}{dx}\left(\frac{dv}{dx}-\frac{du}{dy}\right),$$

where $g^2=\frac{A}{\rho}$, and $\gamma^2=\frac{B}{\rho}$.

These equations might be immediately employed in their present form; but they will take a rather more simple form, by making

$$\left.\begin{aligned} u&=\frac{d\phi}{dx}+\frac{d\psi}{dy}\\ v&=\frac{d\phi}{dy}-\frac{d\psi}{dx}\end{aligned}\right\}\text{.......................(13)};$$

ϕ and ψ being two functions of x, y, and t, to be determined.

By substitution, we readily see that the two preceding equations are equivalent to the system

$$\left.\begin{aligned} \frac{d^2\phi}{dt^2}&=g^2\left(\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}\right)\\ \frac{d^2\psi}{dt^2}&=\gamma^2\left(\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}\right)\end{aligned}\right\}\text{...................(14)}.$$

In like manner, if in the second medium we make

$$\left.\begin{aligned} u_, &= \frac{d\phi_,}{dx} + \frac{d\psi_,}{dy} \\ v_, &= \frac{d\phi_,}{dy} - \frac{d\psi_,}{dx} \end{aligned}\right\} \text{.........................} (15),$$

we get to determine $\phi_,$ and $\psi_,$ the equations

$$\left.\begin{aligned} \frac{d^2\phi_,}{dt^2} &= g_,^2 \left(\frac{d^2\phi_,}{dx^2} + \frac{d^2\phi_,}{dy^2}\right) \\ \frac{d^2\psi_,}{dt^2} &= \gamma_,^2 \left(\frac{d^2\psi_,}{dx^2} + \frac{d^2\psi_,}{dy^2}\right) \end{aligned}\right\} \text{..................} (16),$$

and as we suppose the constants A and B the same for both media, we have

$$\frac{\gamma}{\gamma_,} = \frac{g}{g_,}.$$

For the complete determination of the motion in question, it will be necessary to satisfy all the conditions due to the surface of junction of the two media. But, since $w = 0$ and $w_, = 0$, also, since $u, v, u_,, v_,$ are independent of z, the equations (5) and (6) become

$$u = u_,, \quad v = v_,;$$

$$A\left(\frac{du}{dx} + \frac{dv}{dy}\right) - 2B\frac{dv}{dy} = A\left(\frac{du_,}{dx} + \frac{dv_,}{dy}\right) - 2B\frac{dv_,}{dy},$$

$$\frac{du}{dy} + \frac{dv}{dx} = \frac{du_,}{dy} + \frac{dv_,}{dx},$$

provided $x = 0$. But since $x = 0$ in the last equations, we may differentiate them with regard to any of the independent variables except x, and thus the two latter, in consequence of the two former, will become

$$\frac{du}{dx} = \frac{du_,}{dx}, \quad \frac{dv}{dx} = \frac{dv_,}{dx}.$$

Substituting now for $u, v,$ &c., their values (13) and (15), in ϕ and ψ, the four resulting conditions relative to the surface of junction of the two media may be written,

$$\left.\begin{aligned} \frac{d\phi}{dx}+\frac{d\psi}{dy}&=\frac{d\phi_{,}}{dx}+\frac{d\psi_{,}}{dy}\\ \frac{d\phi}{dy}-\frac{d\psi}{dx}&=\frac{d\phi_{,}}{dy}-\frac{d\psi_{,}}{dx}\\ \frac{d^2\phi}{dx^2}+\frac{d^2\psi}{dx\,dy}&=\frac{d^2\phi_{,}}{dx^2}+\frac{d^2\psi_{,}}{dx\,dy}\\ \frac{d^2\phi}{dx\,dy}-\frac{d^2\psi}{dx^2}&=\frac{d^2\phi_{,}}{dx\,dy}-\frac{d^2\psi_{,}}{dx^2} \end{aligned}\right\}\text{(when } x=0\text{)};$$

or since we may differentiate with respect to y, the first and fourth equations give

$$\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}=\frac{d^2\psi_{,}}{dx^2}+\frac{d^2\psi_{,}}{dy^2};$$

in like manner, the second and third give

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}=\frac{d^2\phi_{,}}{dx^2}+\frac{d^2\phi_{,}}{dy^2},$$

which, in consequence of the general equations (14) and (16), become

$$\frac{d^2\psi}{\gamma^2dt^2}=\frac{d^2\psi_{,}}{\gamma_{,}^2dt^2},\quad\text{and}\quad\frac{d^2\phi}{g^2dt^2}=\frac{d^2\phi_{,}}{g_{,}^2dt^2}.$$

Hence, the equivalent of the four conditions relative to the surface of junction may be written

$$\left.\begin{aligned} \frac{d\phi}{dx}+\frac{d\psi}{dy}&=\frac{d\phi_{,}}{dx}+\frac{d\psi_{,}}{dy}\\ \frac{d\phi}{dy}-\frac{d\psi}{dx}&=\frac{d\phi_{,}}{dy}-\frac{d\psi_{,}}{dx}\\ \frac{d^2\phi}{g^2dt^2}&=\frac{d^2\phi_{,}}{g_{,}^2dt^2}\\ \frac{d^2\psi}{\gamma^2dt^2}&=\frac{d^2\psi_{,}}{\gamma_{,}^2dt^2} \end{aligned}\right\}\text{(when } x=0\text{) .... (17).}$$

If we examine the expressions (13) and (15), we shall see that the disturbances due to ϕ and $\phi_{,}$ are normal to the front of the wave to which they belong, whilst those which are due to ψ,

$\psi_{,}$ are transverse or wholly in the front of the wave. If the coefficients A and B did not differ greatly in magnitude, waves propagated by both kinds of vibrations must in general exist, as was before observed. In this case, we should have in the upper medium

$$\left.\begin{aligned} \psi &= f(ax+by+ct) + F(-ax+by+ct) \\ \text{and} \quad \phi &= \chi_{,}(-a'x+by+ct) \end{aligned}\right\} \ldots\ldots (18);$$

and for the lower one

$$\left.\begin{aligned} \psi_{,} &= f_{,}(a_{,}x+by+ct) \\ \phi_{,} &= \chi_{,}(a_{,}'x+by+ct) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (19).$$

The coefficients b and c being the same for all the functions to simplify the results, since the indeterminate coefficients $a_{,}'a_{,}a'$ will allow the fronts of the waves to which they respectively belong, to take any position that the nature of the problem may reqnire. The coefficient of x in F belonging to that reflected wave, which, like the incident one, is propagated by transverse vibrations would have been determined exactly like $a_{,}'a_{,}a'$, as, however, it evidently $=-a$, it was for the sake of simplicity introduced immediately into our formulæ.

By substituting the values just given in the general equations (14) and (16), there results

$$c^2 = (a^2+b^2)\gamma^2 = (a_{,}^2+b^2)\gamma_{,}^2 = (a'^2+b^2)g^2 = (a_{,}'^2+b^2)g'^2,$$

we have thus the position of the fronts of the reflected and refracted waves.

It now remains to satisfy the conditions due to the surface of junction of the two media. Substituting, therefore, the values (18) and (19) in the equations (17), we get

$$f'' + F'' = \frac{\gamma^2}{\gamma_{,}^2} f_{,}'',$$

$$\chi'' = \frac{g^2}{g_{,}^2}\chi_{,}'';$$

$$-a'\chi' + b(f'+F') = a_{,}'\chi_{,}' + bf_{,}',$$

$$b\chi' - a(f'-F') = b\chi_{,}' - a_{,}f_{,}';$$

where to abridge, the characteristics only of the functions are written.

By means of the last four equations, we shall readily get the values of $F''\chi''f_{\prime}''\chi_{\prime}''$ in terms of f'', and thus obtain the intensities of the two reflected and two refracted waves, when the coefficients A and B do not differ greatly in magnitude, and the angle which the incident wave makes with the plane surface of junction is contained within certain limits. But in the introductory remarks, it was shewn that $\frac{A}{B}$ = a very great quantity which may be regarded as infinite, and therefore g and $g_{\prime}$ may be regarded as infinite compared with γ and $\gamma_{\prime}$. Hence, for all angles of incidence except such as are infinitely small, the waves dependent on ϕ and $\phi_{\prime}$ cease to be transmitted in the regular way. We shall therefore, as before, restrain the generality of our functions by supposing the law of the motion to be similar to that of a cycloidal pendulum, and as two of the waves cease to be transmitted in the regular way, we must suppose in the upper medium

$$\left.\begin{aligned} \psi &= \alpha \sin (ax + by + ct + e) + \beta \sin (-ax + by + ct + e_{\prime}) \\ \text{and} \qquad \phi &= e^{a'x} (A \sin \psi_0 + B \cos \psi_0) \end{aligned}\right\} (20);$$

and in the lower one

$$\left.\begin{aligned} \psi_{\prime} &= \alpha_{\prime} \sin (a_{\prime}x + by + ct) \\ \phi_{\prime} &= e^{-a_{\prime}'x} (A_{\prime} \sin \psi_0 + B_{\prime} \cos \psi_0) \end{aligned}\right\} \text{............} (21),$$

where to abridge $\psi_0 = by + ct$.

These substituted in the general equations (14) and (15), give

$$c^2 = \gamma^2 (a^2 + b^2) = \gamma_{\prime}^2 (a_{\prime}^2 + b^2) = g^2 (-a'^2 + b^2) = g_{\prime}^2 (-a_{\prime}'^2 + b^2),$$

or, since g and $g_{\prime}$ are both infinite,

$$b = a' = a_{\prime}'.$$

It only remains to substitute the values (20), (21) in the equations (17), which belong to the surface of junction, and thus we get

$$bA\sin\psi_0+bB\cos\psi_0+b\alpha\cos(\psi_0+e)+b\beta\cos(\psi_0+e_{\prime})$$
$$=-bA_{\prime}\sin\psi_0-bB_{\prime}\cos\psi_0+b\alpha_{\prime}\cos\psi_0,$$
$$bA\cos\psi_0-bB\sin\psi_0-a\alpha\cos(\psi_0+e)+a\beta\cos(\psi_0+e_{\prime})$$
$$=bA_{\prime}\cos\psi_0-bB_{\prime}\sin\psi_0-a_{\prime}\alpha_{\prime}\cos\psi_0\ \ldots\ldots\ldots(22).$$
$$\frac{1}{g^2}(A\sin\psi_0+B\cos\psi_0)=\frac{1}{g_{\prime}^2}(A_{\prime}\sin\psi_0+B_{\prime}\cos\psi_0),$$
$$\frac{1}{\gamma^2}\{\alpha\sin(\psi_0+e)+\beta\sin(\psi_0+e_{\prime})\}=\frac{1}{\gamma_{\prime}^2}\alpha_{\prime}\sin\psi_0.$$

Expanding the two last equations, comparing separately the coefficients of $\cos\psi_0$ and $\sin\psi_0$, and observing that

$$\frac{g}{g_{\prime}}=\frac{\gamma}{\gamma_{\prime}}=\mu \text{ suppose},$$

we get

$$\left.\begin{aligned}A&=\mu^2A_{\prime},\\ B&=\mu^2B_{\prime},\\ \alpha\cos e+\beta\cos e_{\prime}&=\mu^2\alpha_{\prime},\\ \alpha\sin e+\beta\sin e_{\prime}&=0\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots(23).$$

In like manner the two first equations of (22) will give

$$0=A+A_{\prime}-\alpha\sin e-\beta\sin e_{\prime},$$
$$0=A-A_{\prime}+\frac{a\alpha_{\prime}}{b}+\frac{a}{b}(\beta\cos e_{\prime}-\alpha\cos e),$$
$$0=B+B_{\prime}+\alpha\cos e+\beta\cos e_{\prime}-\alpha_{\prime},$$
$$0=B-B_{\prime}+\frac{a}{b}(\beta\sin e_{\prime}-\alpha\sin e);$$

combining these with the system (23), there results

$$\left.\begin{aligned}0&=A+A_{\prime}\\ 0&=B+B_{\prime}+(\mu^2-1)\alpha_{\prime}\\ 0&=A-A_{\prime}+\frac{a\alpha_{\prime}}{b}+\frac{a}{b}(\beta\cos e_{\prime}-\alpha\cos e)\\ 0&=B-B_{\prime}+\frac{a}{b}(\beta\sin e_{\prime}-\alpha\sin e)\end{aligned}\right\}\ \ldots(24).$$

Again, the systems (23) and (24) readily give

$$\left.\begin{aligned} \alpha \sin e &= -\tfrac{1}{2} \cdot \frac{(\mu^2-1)^2}{\mu^2+1} \frac{b}{a} a_, \\ \alpha \cos e &= \tfrac{1}{2} \cdot \left(\mu^2 + \frac{a_,}{a}\right) a_, \\ \beta \sin e_, &= \tfrac{1}{2} \cdot \frac{(\mu^2-1)^2}{\mu^2+1} \frac{b}{a} a_, \\ \beta \cos e_, &= \tfrac{1}{2} \cdot \left(\mu^2 - \frac{a_,}{a}\right) a_, \end{aligned}\right\} \quad\ldots\ldots\ldots (25);$$

and therefore

$$\frac{\beta^2}{\alpha^2} = \frac{(\mu^2+1)^2 \cdot \left(\mu^2 - \frac{a_,}{a}\right)^2 + (\mu^2-1)^4 \frac{b^2}{a^2}}{(\mu^2+1)^2 \cdot \left(\mu^2 + \frac{a_,}{a}\right)^2 + (\mu^2-1)^4 \frac{b^2}{a^2}} \quad\ldots\ldots\ldots (26).$$

When the refractive power in passing from the upper to the lower medium is not very great, μ does not differ much from 1. Hence, $\sin e$ and $\sin e_,$ are small, and $\cos e$, $\cos e_,$ do not differ sensibly from unity; we have, therefore, as a first approximation,

$$\frac{\beta}{\alpha} = \frac{\mu^2 - \frac{a_,}{a}}{\mu^2 + \frac{a_,}{a}} = \frac{\frac{\sin^2\theta}{\sin^2\theta_,} - \frac{\cot\theta_,}{\cot\theta}}{\frac{\sin^2\theta}{\sin^2\theta_,} + \frac{\cot\theta_,}{\cot\theta}} = \frac{\sin 2\theta - \sin 2\theta_,}{\sin 2\theta + \sin 2\theta_,} = \frac{\tan(\theta - \theta_,)}{\tan(\theta + \theta_,)},$$

which agrees with the formula in Airy's *Tracts*, p. 358*, for light polarized perpendicular to the plane of reflexion. This result is only a near approximation: but the formula (26) gives the correct value of $\frac{\beta^2}{\alpha^2}$, or the ratio of the intensity of the reflected to the incident light; supposing, with all optical writers, that the intensity of light is properly measured by the square of the actual velocity of the molecules of the luminiferous ether.

From the rigorous value (26), we see that the intensity of the reflected light never becomes absolutely null, but attains a minimum value nearly when

* [Airy, *ubi sup.* p. 110.]

$$0 = \mu^2 - \frac{a_{\prime}}{a}, \text{ i.e., when } \tan(\theta + \theta_{\prime}) = \infty,$$

which agrees with experiment, and this minimum value is, since (27) gives $\frac{b}{a} = \mu$,

$$\frac{\beta^2}{\alpha^2} = \frac{(\mu^2-1)^4 \frac{b^2}{a^2}}{4(\mu^2+1)^2\mu^4 + (\mu^2-1)^4 \frac{b^4}{a^4}} = \frac{(\mu^2-1)^4}{4\mu^2(\mu^2+1)^2 + (\mu^2-1)^4} \ldots (28).$$

If $\mu = \frac{4}{3}$, as when the two media are air and water, we get

$$\frac{\beta^2}{\alpha^2} = \frac{1}{151} \text{ nearly.}$$

It is evident from the formula (28), that the magnitude of this minimum value increases very rapidly as the index of refraction increases, so that for highly refracting substances, the intensity of the light reflected at the polarizing angle becomes very sensible, agreeably to what has been long since observed by experimental philosophers. Moreover, an inspection of the equations (25) will shew, that when we gradually increase the angle of incidence so as to pass through the polarizing angle, the change which takes place in the reflected wave is not due to an alteration of the sign of the coefficient β, but to a change of phase in the wave, which for ordinary refracting substances is very nearly equal to 180°; the minimum value of β being so small as to cause the reflected wave sensibly to disappear. But in strongly refracting substances like diamond, the coefficient β remains so large that the reflected wave does not seem to vanish, and the change of phase is considerably less than 180°. These results of our theory appear to agree with the observations of Professor Airy. (*Camb. Phil. Trans.* Vol. IV. p. 418, &c.)

Lastly, if the velocity $\gamma_{\prime}$ of transmission of a wave in the lower exceed γ that in the upper medium, we may, by sufficiently augmenting the angle of incidence, cause the refracted wave to disappear, and the change of phase thus produced in the reflected wave may readily be found. As the calculation is extremely easy after what precedes, it seems sufficient to give the result. Let therefore, here, $\mu = \frac{\gamma_1}{\gamma}$, also e, e_1 and θ as before, then $e_1 = -e$, and the accurate value of e is given by

$$\tan e = \mu\sqrt{\mu^2 \tan^2\theta - \sec^2\theta} - \frac{(\mu^2-1)^2\tan\theta}{\mu^2+1}.$$

The first term of this expression agrees with the formula of page 362, Airy's *Tracts**, and the second will be scarcely sensible except for highly refracting substances.

* [Airy, *ubi sup.* p. 114, Art. 133.]