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OVERVIEWS AND TUTORIALS

On the Logical Status of the Virtual Work Law

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Abstract. The law of virtual work (VWL) is probably the first law in the history of mechanics; it is previous to the one on the lever, though not completely distinct from it. Here I will discuss the logical status of VWL, that is whether it is an autonomous principle or a theorem of some sort of mechanics. The problem is complicated by the fact that up to now no universally recognised expression has been accepted for it. From this article the problematical nature of VWL demonstrability is quite clear when the mechanics does not characterise completely the constraints. Italian schools in the XVIII century, even if we do not take Lagrange into consideration, had an important role, both in the development of VWL and in the discussion of its role.

Key words: Virtual work, History of mechanics, Statics, Mechanics, Italian contribution.

1. Introduction

The law of virtual work (VWL) is probably the first quantitative law in the history of mechanics, even older than the law of the lever according to Duhem [1], though not completely distinct from it. Its utility became obvious only after the publication of Lagrange's *Mécanique analytique* in 1788, in which it was both a theoretical instrument – a mechanical principle – and a method able to solve specific mechanical problems. Since then VWL has become an integral part of all handbooks on mechanics, where, generally, it coexists with other laws, from which it is sometimes derived and from which it sometimes derives.

Today its role in mechanics appears less clear, at least in its application in engineering. In the study of the equilibrium of assemblies of rigid bodies it has become scarcely relevant because the cardinal equations of statics, with constraint reactions as auxiliary unknowns, are considered simpler and more direct. The role of VWL has become instead very important in continuous mechanics, where, joined to the calculus of variations, it is used for developing approximate procedures of solution. But one uses more the mathematical nature of weak formulation of VWL than the ability to deal with the constraints. Notwithstanding these contradictions the fascination for VWL has not changed, and I think it is necessary to try to deepen the knowledge of its essence and its logical status.

In this paper the discussion presumes that mechanics can be given an axiomatic structure. This is surely a limitation because most philosophers of science today think that the axiomatic is not the unique or the best way to organise a physical theory. In my opinion, however, the discussion of a relevant problem in the axiomatic organisation will also help us to understand the role the same problem plays in a different organisation, and it is worthy of being pursued independently of the epistemological positions.

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Assuming an axiomatic structure, to discuss the logical status of VWL means to discuss if it is, or should be, the principle of an autonomous mechanics or a theorem of another type of mechanics. To demonstrate a law, a proposition means, in a certain sense, to take it back to other laws assumed as known, to principles accepted according to a current epistemological conception. Until the development of modern axiomatic theories and their application to physical science by the neopositivists of the XX century, a principle was accepted if it had an intuitive nature of proof, possibly *a priori* founded. Nowadays there is a more liberal conception, and evidence of the principles is not asked for anymore; they must only contain enough logical force not to create contradictions [2]. According to this evolution of epistemology, the way of considering VWL has evolved too. Two different points of view have now the same dignity. From the first VWL is not a principle and it must be proved; from the second it is enough that assuming VWL as an axiom, a complete mechanics will result. In this paper the first point of view, which is still the most common, is mainly explored. It must be noticed however that recently, by reconnecting to the old position of the energetic movement [3], the second alternative has gained prominence [4].

The demonstrability of VWL crashes immediately with the fact that today many formulations exist for it [5], and many mechanical reference theories exist as well (for example, Newtonian, Lagrangian, Eulerian, etc. [6]). This is also true for the systems of material points, even though some interesting axiomatisations exist [7–9]. One of the major problems occurring in the different formulations of VWL is the role of the constraints and the constraint reactions. Before the XVIII century the constraints had simply been dealt with as passive elements able to suffer but not react. Only after studies on elasticity and after having accepted models of matter based on interacting corpuscles, one started thinking about constraints as things able to give out force.

This difficulty in including the constraint reactions in a consistent mechanical theory brought about the theoretical development of VWL, which offers a criterion of equilibrium without permitting the intervention of these undesirable forces. However, the proposition which asserts that ‘the sum of forces, each multiplied by the displacement of the point to which it is applied, following the direction of this force, will always be zero’, had always created some problems because of its scarcely intuitive nature; the following comment by Fossombroni [10] is interesting:

> Quella comune facoltà di primitiva intuizione, per cui ognuno si convince facilmente di un semplice assioma geometrico, come per esempio, *che il tutto sia maggiore della parte*, non serve certamente per convenire della sopraccennata verità meccanica, la quale è tanto più complicata di quello che sia uno degli ordinari assiomi, quanto il genio di quei grandi Uomini, che l’hanno ammessa per assioma, supera l’ordinaria misura dell’ingegno umano; ed è in conseguenza necessario per coloro che non ne restano appagati, il procurarsene una dimostrazione dipendentemente da estranee teorie [. . .] ovvero riposarsi sulla fede d’uomini sommi, disprezzando l’usuale ripugnanza ad introdurre in Matematica il peso dell’autorità.[1]

[1] That common faculty of primitive intuition, for which everyone is easily convinced of a simple geometrical axiom such as for example that the whole is greater than the part, is certainly not necessary to agree on the above-mentioned truth of mechanics, which is more complicated than that of ordinary axioms the more the genius of those great men that admitted it as an axiom exceeds the ordinary measure of human intelligence, and is therefore necessary, for those that are not satisfied by it, to obtain a demonstration independent of other theories [. . .] that is to rest upon the faith of great men, despising the usual repugnance to introduce the weight of authority in Mathematics (Fossombroni, Memoria sul principio delle velocità virtuali, p. 13).

After the success of Lagrange's *Mécanique analytique* in 1788 [11], which assumes VWL as the principle of whole mechanics, a lively debate started, and many attempts were made to demonstrate VWL [12]. In the present work I will link up to these attempts and also previous ones and will try to explain in what sense VWL can be demonstrated. The attempts can be divided in two categories. The first one, which I will refer to as fundamental, tries to deduce VWL as a criterion of equilibrium without assuming pre-existent criteria. The second category, which I will refer to as reductionist, tries to deduce VWL from a pre-existent criterion of a pre-existent mechanics.

Vincenzo Riccati, Vincenzo Angiulli, Lazare Carnot [13–15], and Lagrange [12] made attempts in the first direction. The first two thought they could demonstrate VWL using a reference mechanics of a Leibnizian type, lacking however a pre-existent criterion of equilibrium, and with considerations of a metaphysical nature. Lagrange took VWL back to one of its more specific cases that he considered obvious: the law of the pulley. Carnot tried to get to VWL starting from the law of shock using a reference mechanics 'without' force.

Attempts in the second direction were made by French scientists of École Polytechnique, who synthesising and using Lagrange's mechanics categories, used the reference mechanics that derived from the law of lever and the rule of parallelogram. In my opinion the most interesting results are the ones that Fourier [16] and Poinsot [17] obtained. The first one reduces VWL to the law of lever, with a series of auxiliary and not particularly severe assumptions, and therefore successfully. The second one develops a complete mechanical theory – that includes the constraints – based on the rule of parallelogram.

Poinsot's demonstrations is the one that has influenced the most the subsequent treatises on statics. Its prevalence over Fourier's, which may be more interesting, derives from the nature of the reference mechanics considered, based on the rule of parallelogram, according to which the laws of equilibrium can be reduced more easily to mathematical formulas.

In recent scientific literature the problem of VWL's role is faced only in handbooks, where the author reports his idea on the subject in a few pages, usually going back to a limited number of basic versions. Regarding this refer to Drago [5] where a large number of texts are studied.

There are not many recent works of a theoretical nature on the basic aspects of VWL, because there have not been recent attempts at funding studies on the mechanics of the material point, even if the subject is far from being fully understood. It also seems that there are very few recent studies of a historical nature on VWL [18]; it is dealt with only marginally in the numerous monographs on Lagrange, Laplace, etc.

In the present paper I will consider those aspects indicated as being reductionist, my aim being to fill in a small part of what I believe is missing from the knowledge of the logical status of VWL. On the one hand, I will try to underline the logical problems, and precisely because a generally accepted formulation of classical mechanics does not exist, the situation is not considered in its general framework. To be exact, I presume a type of Newtonian mechanics with only material points and forces applied to them. The formulation of VWL will emerge in a natural way, and a possible extension to more general situations may only create technical complications. On the other hand, I will illustrate some attempts to demonstrate VWL at the end of the XVIII century and the beginning of the XIX century.

2. The Theorem and the Principle of Virtual Work

To start with, I consider a reference mechanics of the Newtonian type, hereafter referred to as the N_0 mechanics, in which only forces and material points exist. In this mechanics a system S, of a finite number of material points, is said to be constrained when the configuration C of

S is described by a differentiable submanifold M of dimension $m < 3n$ instead of a $3n$ dimensional differentiable manifold N, necessary in principle to describe the configuration of S.

External forces act on the material points, of which the law of temporal and spatial variation is assigned, and other forces, associated with constraints, not known beforehand. The forces of constraints, hereafter named constraint reactions, are simply the ‘ordinary’ forces necessary to maintain the constraints; only they are unknown. Collecting known external forces in the vector $\mathbf{f}$ and the constraint reactions in the vector $\mathbf{r}$, I assume the following axiom of equilibrium:

AXIOM 1. *A configuration* C *of a system of material points constrained to move on a differentiable manifold* M *is a configuration of equilibrium if and only if the equation can be satisfied each time t*:

$$\mathbf{f} + \mathbf{r} = 0$$

In the following, for the sake of simplicity, one *given configuration* C *and* all *times t will always implicitly be concerned.*

Define now the virtual work of the forces acting on S as the linear form on the $3n$-dimensional vector space V_{N} associated to N: $L(\mathbf{v}) = (\mathbf{f} + \mathbf{r}) \cdot \mathbf{v}$, where dot means inner product and $\mathbf{v}$ is any vector of V_{N}, called virtual displacement. Consider also the other two linear forms $L_{\mathrm{f}}(\mathbf{v}) = \mathbf{f} \cdot \mathbf{v}$ and $L_{\mathrm{r}}(\mathbf{v}) = \mathbf{r} \cdot \mathbf{v}$ called respectively virtual work of the active forces and virtual work of the constraint reactions. One can easily demonstrate the following theorem of virtual work:

THEOREM 1. *A system of material points constrained to move on a differential manifold* M *is in a state of equilibrium if and only if* $L_{\mathrm{f}}(\mathbf{v}) + L_{\mathrm{r}}(\mathbf{v}) = 0$, *for every* $\mathbf{v}$ *belonging to* V_{N}.

To verify the equilibrium, under Axiom 1 or with Theorem 1, one has to be able to specify the way the constraint reactions depend on the manifold M and eventually on other parameters that define the system of material points. A traditional way to characterise the constraint reactions is to introduce the concept of smooth constraints. By considering the m-dimensional vector space $\mathrm{T_C(M)}$ tangent to M at C, whose vectors $\mathbf{u}$ are still called virtual displacements, the following definition is given:

DEFINITON 1. A system of constraints associated to the manifold M and to a system of material points S is smooth if, and only if, for every configuration C of S, $L_{\mathrm{r}}(\mathbf{u}) = 0$, for every $\mathbf{u}$ belonging to $\mathrm{T_C(M)}$.

For smooth constraints, the following theorem can easily be proved, from Theorem 1:

THEOREM 2. *IF the constraints are smooth THEN a system of material points constrained to move on a differentiable manifold* M *is in a state of equilibrium if and only if* $L_{\mathrm{f}}(\mathbf{u}) = 0$, *for every* $\mathbf{u}$ *belonging to* $\mathrm{T_C(M)}$.

Notice that sometimes vectors $\mathbf{u}$ *of* $\mathrm{T_C(M)}$ are also called virtual velocities. This is justified because any vector $\mathbf{u}$ can be considered the tangent to a path of M passing through C; if this path is parameterised with a (virtual) time, the velocity of the virtual motion along it is parallel to $\mathbf{u}$. When adopting this definition one should refer to $\mathbf{f} \cdot \mathbf{u}$ as ‘virtual power’ instead of ‘virtual work’. Nowadays the use of one or another term is only a question of style. I adopt the second nomenclature because it is more common; in the past, however, virtual velocities and virtual displacements had some differences and consequently produced different approaches (see Section 3).

Theorem 2 is also referred to as theorem of virtual work, as Theorem 1. Moreover, if one does not specify, Theorem 2 is the *Theorem of virtual work* itself. It will then appear that the problem of the logical status of VWL is solved: when properly formulated, it is a theorem of statics. Unfortunately that is no more than an illusion masked by the words in which the concept of smooth constraint has been given. Actually there is not an operative criterion to establish whether or not a constraint is smooth, because it is so if and only if $L_{\mathrm{r}}(\mathbf{u}) = 0$ and we cannot evaluate $L_{\mathrm{r}}(\mathbf{u})$ because we do not know $\mathbf{r}$. So Definition 1 gives the circularity: for a smooth constraint $L_{\mathrm{r}}(\mathbf{u}) = 0$, *if* $L_{\mathrm{r}}(\mathbf{u}) = 0$ the constraint is smooth; and Theorem 2 is useless.

To legitimate the usefulness of the theorem of virtual work and therefore the opportunity of referring to Theorem 2 as to VWL, one needs an operative criterion to establish *a priori* if a constraint is smooth or not. A way to use Theorem 2 is that of enlarging the N_0 mechanics by adding a statement about constraints, which should take the form of the following axiom:

AXIOM 2. *(All) The constraints are smooth.*

Therefore from Axiom 2, applying the modus ponens to Theorem 2, the following theorem is obtained:

THEOREM 3. *A system of material points constrained to move on a differentiable manifold* M *is in a state of equilibrium if and only if* $L_{\mathrm{f}}(\mathbf{u}) = 0$ *for every* $\mathbf{u}$ *belonging to* $\mathrm{T_C(M)}$.

Theorem 3 is usually called the *Principle of virtual work* (VWP), for historical reasons, independent of its being considered a theorem or not. Notice however that Theorem 3 is not a theorem of the N_0 mechanics because it derives from Axiom 1 of N_0 and from Axiom 2 surely independent of N_0 (Theorem 3 could be a theorem of N_0 only if Axiom 2 were its theorem. But this is not the case). Because of the critical role it plays in the proof of VWP, Axiom 2 itself is often called the principle of virtual work. When Axiom 2 is interpreted as a form of virtual work principle, we can say that VWP (Theorem 3) can be proved if and only if VWP (Axiom 2) can be proved. However, I will not accept this definition, and with the term VWP I will be always referring to Theorem 3.

Before discussing further the problem of demonstrability of Theorem 3 (or Axiom 2), it is convenient to see whether or not Theorem 2 can be utilised by enlarging, only a little, the N_0 mechanics up to the N_1 mechanics. What more is it possible to say about the constraints for the corpuscular N_0 mechanics? This mechanics permits, with a change of perspective, to study the interaction of 'bodies' with the system S whose equilibrium is considered. In N_1, bodies of the empirical world are presumed created by material points that work as centres of forces. Then a constraint, more than to an algebraic equation, can be associated, as normally it is, to a set of bodies that are 'hard' enough to be considered impenetrable. When a material point of S gets close to a body, some forces are awakened – the constraint reactions – that oppose to penetration of this by that. Knowing the laws of the centres of forces depending on the distance, the laws of interaction between the constraint-body and the material point, that is the constitutive relationship of the constraint system, can be determined.

In this way there will not be any problems in deciding if a particular constraint is smooth ($L_{\mathrm{r}}(\mathbf{u}) = 0$) or not on the basis of its constitutive relationship and on Definition 1. The theorem of virtual work Theorem 2 will then make sense because an operative criterion will exist to apply it in each case on the basis of considerations of empirical nature. One must notice, though, that in mechanics, one tends to apply the principle and not the theorem of virtual work because, generally, the hypothesis of smooth constraints is not subjected to investigation because in practice it is not possible, and therefore when we speak of VWL we are actually referring to VWP. Therefore the problem of demonstrability of VWP (Theorem 3) must be considered.

It has been seen above that by introducing Axiom 2, Theorem 3 becomes a theorem of an enlarged mechanics. But it was and is still felt that this is not a good move, because Axiom 2 is not self-evident. Admitting that the constraints are formed by bodies, in the past it was thought of demonstrating Axiom 2 assuming first an axiom that is weaker. Empirical experience suggests that a surface of a body polished and possibly oiled, which in current language is called smooth (2), is smooth (1) according to Definition 1, because it is found that VWP gives correct results for it (notice however that smooth$_{(1)}$ is a nominal definition (*quid nominis*) while smooth$_{(2)}$ is a real definition (*quid rei*) and from a logical point of view smooth$_{(1)}$ has nothing to do with smooth$_{(2)}$). People are then justified in assuming the following axiom:

AXIOM 3. *IF the surface of a body is smooth*$_{(2)}$ *THEN it is a smooth*$_{(1)}$ *constraint for material points.*

Axiom 3 is weaker than Axiom 2 because it contains a condition for smoothness$_{(1)}$ and mainly because M is not usually a surface of the three-dimensional space owing to the internal constraints (think for example of two material points constrained to maintain constant their distance). It is clear that this axiom expresses an ideal; practically the constraints will never be smooth$_{(2)}$ because of the inevitable irregularities. When a surface is not smooth$_{(2)}$ it can still be assumed as smooth$_{(1)}$, but results obtained by applying VWP should be regarded only as 'approximated'. Is it possible to demonstrate Axiom 3 in the N_1 mechanics? There are doubts. The only possibility to prove Axiom 3 lies in the criteria of symmetry and sufficient reason, by affirming that reactions must always be orthogonal to the constraints because there is no reason for them not to be. But in the corpuscular mechanics N_1 the same concept of surface of a body presents some difficulties. Even if one should ignore this aspect, the demonstration of normality of the reaction to the surface of the constraint (necessary and sufficient condition for $L_r(\mathbf{u}) = 0$) will require the use of axioms on corpuscle forces with a nature of evidence hardly superior to Axiom 3's. Thus Axiom 3 cannot be possibly proved in N_1; consequently it should be considered an independent axiom justified only by the empirical evidence.

But even if we admit Axiom 3, it is not possible to demonstrate Axiom 2. In fact, Axiom 3 does not permit us to say anything about the internal constraints; in particular it does not permit us to say anything about the rigid body constraints, for which the smoothness concept does not appear to be very intuitive, because no movement is possible among internal particles. So, if one does not change the reference mechanics, even if Axiom 3 is admitted, Axiom 2 is not demonstrable, and therefore VWP is not a theorem. A possible extension of the axiom of mechanics of reference, following Euler, can consist in substituting Axiom 1 with the cardinal equations of statics (N_2 mechanics). In this way one will be advantaged in dealing with the applicability of VWP to the rigid body. But in the presence of internal constraints among the different rigid bodies or of external constraints in rigid bodies, the difficulties encountered in dealing with the system of material points will present themselves again. Concluding, when in a reference mechanics, like N_0, N_1 *or* N_2, there are not assumptions of an empirical nature dealing with the constraints, creating the problem of demonstrability of VWL makes no sense.

To understand the difficulty in proving Axiom 2 from Axiom 3, it is very interesting to analyse Poinsot's reasoning referred to in the subsequent paragraphs. Poinsot succeeded in demonstrating Axiom 2 from Axiom 3 but only at the price of enlarging the classical framework of statics with some very questionable 'principles', which allow us to characterise completely the constraint reactions.

3. The Historical Evolution of Virtual Work Law

3.1. A Short Summary

From Greek origins of mechanics till now, there have been two formulations of VWL. The first one dates back to the Aristotelian school, and today it will go under the name of law of virtual velocities. The second one may be clearly found only with Jordanus de Nemore (XIII century), but probably it was already known in Hellenistic times, and today it will go under the name of law of virtual displacements.

Johann Bernoulli, at the beginning of the XVII century [19, 20], formulated VWL with the concept of infinitesimal virtual displacement, and unified the Aristotelian and de Nemore points of view. Lagrange gave VWL its modern form a half century later, in 1764 [21]:

> C'est un principe généralement vrai en Statique que, si un système quelconque de tant de corps ou de points que l'on veut, tirés chacun par des puissances quelconques, est en équilibre, et qu'on donne a ce système un petit mouvement quelconque, en vertu duquel chaque point parcoure un espace infiniment petit, la somme des puissances, multiplies chacune par l'espace que le point ou elle est appliquée parcourt suivant la direction de cette même puissance, sera toujours égale à zéro.[2]

Both Bernoulli and Lagrange introduced VWL without any proof.

The appearance of the first edition of Lagrange's *Mécanique analytique* [11], with the importance that it gave to VWL (yet not proved), was the occasion for a lively discussion on its logical status, and it was also the occasion for a critical analysis of the principles of mechanics. The importance of this analysis, which does not have any precedence in the history of classical mechanics, is not understood today. The list of scientists interested in the problem indicates the efforts made and the possibility of learning much by following their ideas: L. Carnot, Lagrange, Laplace, Poinsot, Fourier, Prony, Ampère and subsequently also Cauchy, Gauss, Poisson, and Ostrogradsky. A synthesis of the ideas developed by some of these scientists is reported in Bailhache's book [18].

According to the essentially Aristotelian epistemology of the time, VWL could not be accepted as a principle because it was not evident *a priori*; it had to be demonstrated, or reduced to a theorem of another mechanics approach, or it would have been supplied by a more convincing version of it.

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[2] A principle generally true in statics exists according to which if a system of how many bodies points is wanted, forced each by arbitrary forces, is in an equilibrium state and if somebody gives the system a small motion, arbitrary, so that each point covers an infinitesimal space, the summation of forces, each multiplied by the space covered from the point it is applied, is always equal to zero (Lagrange, Recherches sur la libration de la Lune, pp. 8–9, Ouvres, Tome VI).

The problem of the demonstration of VWL provoked a strong discussion, especially in France. In Italy too there were important contributions. Before the publication of the *Mécanique*, one must remember Vincenzo Riccati [13], Vincenzo Angiulli [14] and François Daviet de Foncenex [22], and after its publication Vittorio Fossombroni's contribution must be quoted too. In Italy the role of VWP continued to be relevant also in the XIX century and at the beginning of the XX century, see for example works by Levi Civita [23] and Signorini [24]. In the following pages, having regard for space and the need for internal consistency, I will describe shortly Riccati's and Angiulli's reasoning, which is foundational; then I will describe briefly Fossombroni's work, which follows a reductionist view, and then devote much space to Poinsot, again a reductionist view.

3.2. The Italian School

Vincenzo Riccati and Vincenzo Angiulli presented a version of VWL that went under the name of Principle of actions [13, 14]. Although the idea is manly Riccati's, I will explain only Angiulli's reasoning, which is less original but more careful as regards the foundational aspects. Angiulli tried to deduce VWL not starting from other principles of mechanics, but from 'indubitable' metaphysical principles, among which is the equivalence of the cause with the effect. He starts from the Leibnizian concept of dead force, which he presents as an infinitesimal impulse, of $f\ \mathrm{d}s$ type (where f is the intensity of the impulse, identified with the dead force, and $\mathrm{d}s$ is the infinitesimal displacement of the point which the dead force is applied to), continuously renovated because of the effect of gravity or other causes and continuously destroyed by the action of constraints. Once the constraints are removed, the impulses can be accumulated, and the action of dead force consists in the cumulative effect of the impulses that are not destroyed by the constraints. It generates the *live force*. In the initial instant the action is infinitesimal, but it differs from dead force because the infinitesimal $\mathrm{d}s$ is different from zero. With the introduction of the infinitesimal action, Angiulli can state his principle of actions, which he qualifies as a theorem because it is demonstrated by metaphysical considerations:

> L'equilibrio nasce da ciò, che le azioni delle potenze, che equilibrar si devono, se nascessero, sarebbero uguali, e contrarie; e perciò l'uguaglianza, e la contrarietà delle azioni delle potenze è la vera causa dell'equilibrio [...]. L'equilibrio non è altro, che l'impedimento de' moti, cioè degli effetti dell'azione delle potenze, a cui non è meraviglia se corrisponde l'impedimento delle cause, cioè delle azioni stesse.[3]

The principle of action implicates the relation $\sum f_\mathrm{i}\,\mathrm{d}s_\mathrm{i} = 0$, where $f_\mathrm{i}\,\mathrm{d}s_\mathrm{i}$ are the elementary actions that develop in the infinitesimal displacements $\mathrm{d}s_\mathrm{i}$ compatible with the constraints. Therefore it is a possible formulation of VWL. In Angiulli's treatment, the status of the constraints is that of hard bodies, that is of idealised bodies that absorb all the impulses, both of dead force and of live force, in the direction in which they act; they do not have any effect on the impulses in the directions where motion is permitted. That is, the constraints obey an economy criterion, acting only for as much as they are respected. One should notice that the constraints have only the effect of destroying the motions and that they do not produce any constraint force because this concept is unknown to Leibnizian mechanics.

[3] Equilibrium is born from what, if the actions of powers, which must equilibrate themselves, were born, they would be equal and contrary; then the equality, and the opposition of powers is the true cause of equilibrium [...]. Equilibrium is nothing but the impediment of motions, that is of the effects of the action of powers, which is no surprise if the impediment of the cause, that is the actions themselves, corresponds to it (Angiulli, Discorso intorno agli equilibri, p. 17).

Less clear is the role played by Daviet de Foncenex (born in Savoia in 1734, died in Casale in 1799) who was Lagrange's student, though older than him. In 1762 he wrote the paper *Sur les principes fondamentaux de la Mécanique* in the second volume of *Miscellanea philosophica mathematica Societatis privatae taurinensis*. In the same volume appeared two basic papers by Lagrange regarding the calculus of variations and its application to mechanics [25]. Foncenex's paper contains many interesting statements concerning VWL, one of which says that the principle of virtual work 'can be considered with reason as the most fruitful and universal of whole mechanics; all others reduce to it without effort [...] they are nothing but the same principle reduced in formulas'. These are the same words Lagrange will use in its *Mécanique analytique*. The opinion of many historians, among whom is Galletto [25], is that Foncenex's paper was strongly influenced by Lagrange. So studying Foncenex is important only to understand the evolution of Lagrange's mind but not to understand the evolution of VWL itself; for this reason he will not be dealt with here.

Vittorio Fossombroni demonstrates, in his work in 1794, VWL in the case of a rigid body, not constrained. His demonstration of VWL should be considered the first public convincing reduction of Bernoulli's principle to the cardinal equations of statics. Fossombroni's demonstration, translated in vector calculus, is very simple. The virtual displacements associated to a generic act of infinitesimal rigid motion are supplied by the relation $\mathrm{d}\mathbf{s}_\mathrm{i} = \boldsymbol{\omega} \times \mathbf{r}_\mathrm{i} + \mathrm{d}\mathbf{s}_0$, where $\boldsymbol{\omega}$ and $\mathrm{d}\mathbf{s}_0$ are arbitrary vectors, $\mathbf{r}_\mathrm{i}$ is the position vector of the point of application of the generic force $\mathbf{f}_\mathrm{i}$, and $\times$ is the symbol of the vector product. The virtual work of the forces applied to the rigid body is therefore given by $L = \Sigma\mathbf{f}_\mathrm{i} \cdot \mathrm{d}\mathbf{s}_\mathrm{i} = \Sigma\mathbf{f}_\mathrm{i} \cdot (\boldsymbol{\omega} \times \mathbf{r}_\mathrm{i} + \mathrm{d}\mathbf{s}_0) = (\Sigma\mathbf{r}_\mathrm{i} \times \mathbf{f}_\mathrm{i}) \cdot \boldsymbol{\omega} + (\Sigma\mathbf{f}_\mathrm{i}) \cdot \mathrm{d}\mathbf{s}_0$, where the permutable properties of the mixed product is considered. With these positions to prove VWL is easy: (1) from the cardinal equations of statics $\Sigma\mathbf{f}_\mathrm{i} = 0$, $\Sigma\mathbf{r}_\mathrm{i} \times \mathbf{f}_\mathrm{i} = 0$, that hold the equilibrium, it follows $L = 0$; (2) vice versa, because $\boldsymbol{\omega}$ and $\mathrm{d}\mathbf{s}_0$ are arbitrary, $L = 0$ implies the cardinal equations of statics. The effective proof presented by Fossombroni is much more elaborate, but only because he did not know the vector calculus, which had yet to be developed.

Fossombroni's attempt to substitute infinitesimal displacements, with which he had some embarrassment, with displacements of arbitrary entity is quite interesting. He qualifies VWL with the term 'law of moments', following Galileo's and Lagrange's terminology, when he must use the infinitesimal virtual displacement, and with the term 'law of forces', when he can use the finite displacements. Fossombroni demonstrated that if the forces are parallel to each other and their application points are aligned, then the virtual work of these forces is zero for any finite rigid motion, and therefore both the law of force and the law of moments are valid. Fossombroni's idea was generalised to the case of forces in the space with application points on a surface by Poinsot, who felt the same awkwardness in the use of infinitesimal quantities.

3.3. The Contribution by Poinsot

Louis Poinsot, with Fourier, was the one who was the most successful in reductionist attempts. But, as I said before, because his mechanics of reference based on the rule of parallelogram was, and still is, considered by mathematicians and physicists more interesting than Fourier's based on the law of lever, Poinsot's formulation of VWL [17] has become a model of demonstration for almost all the handbooks of statics.

Poinsot takes for granted Axiom 3, or more precisely a modified version of it, which does not require the concept of constraint reaction, which can be stated in the form: a material point is in equilibrium on a surface if and only if the force applied is orthogonal to it:

> En effet, on démontre que, si un point n'a d'autre liberté que celle de se mouvoir sur une surface ou sur une ligne courbe fixement arrêtée; il n'y peut être équilibre, à moins que la résultantes des forces qui le sollicitent ne soit perpendiculaire à cette surface où a cette ligne courbe.[4]

In reality he does not demonstrate what he declares, and does not even have the possibility to do so, because his constraints are not beings but only analytical expressions. Poinsot is not the only one to hold as self-evident that a constraint cannot oppose itself to tangent forces; Laplace is convinced of this too:

> Or la force de pression d'un point sur une surface lui est perpendiculaire; autrement elle pourrait se décomposer en deux, l'une perpendiculaire à la surface, et qui serait détruite par elle, l'autre parallèle à la surface, et en vertu de laquelle le point n'aurait point d'action sur cette surface, ce qui est contre la supposition.[5]

The reasoning is not conclusive; in fact it reduces itself to a trivial tautology according to which the constraint does not act in a tangent direction because it does not act in a tangent direction. Lagrange expresses similar ideas too:

> Or si l'on fait abstraction de la force P, et qu'on suppose que le corps soit forcé de se mouvoir sur cette surface, il est claire que l'action, ou plutôt la résistance que la surface oppose ou corps, ne peut agir que dans une direction perpendiculaire à la surface;[6]

But he seems to realise the difficulty of the problem because often in his essays he takes the constraints expressed by an analytic expression back to the constraints created with inextensible and non-rigid wires. In this case the orthogonality of the reaction to the surface, for example to the spherical surface described by a material point, appears more convincing, although basically it is taken back to our everyday experience.

Beyond the Axiom 3, Poinsot considers other principles; the first Poinsot presents is the 'solidification' principle, according to which if other constraints are added – both internal and external – to a system of bodies in equilibrium, the equilibrium is not altered, and the distribution of the forces in the points that have not been constrained is not modified. This principle had been used by Stevin, Clairaut, and Euler to study fluids and will be used in the future by Duhem [26] to obtain field equations of equilibrium for a tridimensional continuum. The second principle is presented by Poinsot as the fundamental property of equilibrium; it states that the necessary condition – and sufficient too – for equilibrium of a free system is

[4] Actually it is demonstrated that, if a point does not have any freedom in space other than moving over a surface or along a curve line strongly fixed, there cannot be any equilibrium unless the resultant of the forces that influence it is not perpendicular to this surface or to this curve line (Poinsot, Théorie générale de l'équilibre et du mouvement des systèmes, p. 234).

[5] Now the force of the pressure of a point on a surface is perpendicular, otherwise it could divide itself in two [components], one perpendicular to the surface, that would be destroyed by it, and the other parallel to the surface, and because of it, the point would not act at all on the surface, that would be against the assumption (Laplace, Mécanique Celeste, p. 11).

[6] If one abstracts the force P, and supposes that a body is forced to move on this surface, it is clear that the action, or rather the surface's resistance to the body, cannot act in the direction orthogonal to the surface (Lagrange, Théorie des fonctions analytique, Œuvre, Vol. 9 (1813), p. 378).

that all forces applied to various points should be able to reduce themselves to any number of couples of forces identical and opposite to each other. The third principle is required by the second one and deals with the possibility of decomposing a force in other forces using the parallelogram rule. A fourth principle of mechanics deals with the superposition of the effects of the constraints, according to which if more constraints act on a system of points, they are able to absorb the sum of the forces that each constraint is able to absorb separately. On the basis of these principles, which are however arguable, he demonstrates Axiom 2. More precisely he proves that reactions are 'orthogonal' to constraints:

> *Quelles que soient les équations qui règnent entre les coordonnées des différens points du système, chacune d'elles, pour l'équilibre, demande q'on applique à ces points, le long le leurs coordonnées, des forces quelconques proportionnelles aux fonctions primes de cette équation, relativement à ces coordonnées respectives* (in italic in the original paper).
>
> Ainsi, en représentant par $L = 0$, $M = 0$, &c. des équations quelconques entre les coordonnées x, y, z; x', y', z', &c. des différens points, et par λ, μ &c. des coefficients quelconques indéterminés, on aura, pour les forces totales, X, Y, Z; X', Y', Z'; &c., qui doivent être appliquées à ces points suivant leurs cordonnées:
>
> $$X = \lambda\left(\frac{\mathrm{d}L}{\mathrm{d}x}\right) + \mu\left(\frac{\mathrm{d}M}{\mathrm{d}x}\right) + \&c. \qquad Y = \lambda\left(\frac{\mathrm{d}L}{\mathrm{d}y}\right) + \mu\left(\frac{\mathrm{d}M}{\mathrm{d}y}\right) + \&c.$$
> $$Z = \lambda\left(\frac{\mathrm{d}L}{\mathrm{d}z}\right) + \mu\left(\frac{\mathrm{d}M}{\mathrm{d}z}\right) + \&c.$$
> $$X' = \lambda\left(\frac{\mathrm{d}L}{\mathrm{d}x'}\right) + \mu\left(\frac{\mathrm{d}M}{\mathrm{d}x'}\right) + \&c. \qquad Y' = \lambda\left(\frac{\mathrm{d}L}{\mathrm{d}y'}\right) + \mu\left(\frac{\mathrm{d}M}{\mathrm{d}y'}\right) + \&c.$$
> $$Z' = \lambda\left(\frac{\mathrm{d}L}{\mathrm{d}z'}\right) + \mu\left(\frac{\mathrm{d}M}{\mathrm{d}z'}\right) + \&c. \qquad \&c.$$
>
> Si l'on élimine de ces équations les indéterminées λ, μ &c., il restera les conditions de l'équilibre proprement dites, c'est-à-dire, les relations qui doivent avoir lieu entre les seuls forces appliquées et les coordonnées des leurs points d'application pour l'équilibre du système.[7]

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[7] *Whatever are the equations among the coordinates of the different points of the system, for the equilibrium, each one requires that, along the coordinates, any force should be applied respectively to these points proportionally to the derivative of these equations with respect to the coordinates.*

Therefore, representing with $L = 0$, $M = 0$, etc. the equations of any coordinates x, y, z, x', y', z', etc. of the different points, and with λ, μ, etc. any undetermined coefficients for the forces X, Y, Z, X', Y', Z', etc. that must be applied to these points according to the coordinates, there will be:

$$X = \lambda\left(\frac{\mathrm{d}L}{\mathrm{d}x}\right) + \mu\left(\frac{\mathrm{d}M}{\mathrm{d}x}\right) + \&c. \qquad Y = \lambda\left(\frac{\mathrm{d}L}{\mathrm{d}y}\right) + \mu\left(\frac{\mathrm{d}M}{\mathrm{d}y}\right) + \&c.$$
$$Z = \lambda\left(\frac{\mathrm{d}L}{\mathrm{d}z}\right) + \mu\left(\frac{\mathrm{d}M}{\mathrm{d}z}\right) + \&c.$$
$$X' = \lambda\left(\frac{\mathrm{d}L}{\mathrm{d}x'}\right) + \mu\left(\frac{\mathrm{d}M}{\mathrm{d}x'}\right) + \&c. \qquad Y' = \lambda\left(\frac{\mathrm{d}L}{\mathrm{d}y'}\right) + \mu\left(\frac{\mathrm{d}M}{\mathrm{d}y'}\right) + \&c.$$
$$Z' = \lambda\left(\frac{\mathrm{d}L}{\mathrm{d}z'}\right) + \mu\left(\frac{\mathrm{d}M}{\mathrm{d}z'}\right) + \&c. \qquad \&c.$$

The next step, to demonstrate VWP, which may appear trivial according to the considerations of the first part of this article, did not present itself in these terms to Poinsot simply because the then dominant idea of virtual displacement was based on the concept of infinitesimals, and this created some awkwardness. Poinsot was the first one to eliminate the concept of infinitesimal displacement and to substitute it with the concept of virtual velocity. To understand Poinsot's hostility, shared by others, towards virtual infinitesimal displacements one should consider the fact that the idea of infinitesimal was still obscure and there was no agreement on it. Moreover, Poinsot believed that virtual velocity, which had to be used instead of virtual displacements, should refer to virtual time and not to real one (the problem presents itself in the case where the forces depend on time and/or velocity). According to Poinsot, real time and virtual time run in different universes: 'the displacement [...] is a simple change of position in which time has no importance'.

The following statements illustrate Poinsot's position regarding virtual displacements:

> On voit par-là qu'on peut prendre des vitesses quelconques finies [...]. Quand l'on veut mesurer ces vitesses par les espaces mêmes que les corps décrivent réellement, comme elles, varient à chaque instant par la liaison des corps, il faut prendre ces espaces, infiniment petits, sans quoi ils ne mesureraient plus les vitesses imprimées; et c'est ainsi qu'on tombe dans les vitesses virtuelles proprement dites, où le principe vient perdre une partie de sa clarté.[8]

> Le principe de cette manière n'offre plus aucune trace dé ces idées de mouvemens infiniment petits, et de perturbation d'équilibre, qui paraissent étrangères à la question, et qui laissent dans l'esprit quelque chose d'obscur.[9]

Substituting virtual displacement with virtual velocity, the demonstration of VWP is immediate. I will report only the necessary part – according to which if the system is balanced then the virtual work of the force is zero – with Poinsot's words and symbols:

> Soit le système défini par les équations suivantes, entre les coordonnées des corps,
>
> $$f(x, y, x; x', y', z', \&c.) = 0. \qquad \phi(x, y, x; x', y', z', \&c.) = 0. \qquad \&c. \qquad \text{(A)}$$
>
> Supposons qu'on imprime à tous ces corps des vitesses quelconques qu'ils puissent avoir actuellement sans violer les conditions de la liaison; les coordonnées $x, y, z; x', y', z'$ &c., varieront avec le temps t, dont il faudra les regarder comme fonctions; et, pour que les vitesses imprimées $(\mathrm{d}x/\mathrm{d}t)$, $(\mathrm{d}y/\mathrm{d}t)$, $(\mathrm{d}z/\mathrm{d}t)$, $(\mathrm{d}x'/\mathrm{d}t)$, &c. soient permises par liaison

If the indeterminate λ, μ, etc., are eliminated from these equations, they will remain the condition of equilibrium properly said, that is the relations that there should be among the only forces applied and the coordinates of their points of application (Poinsot, Théorie générale de l'équilibre et du mouvement des systèmes, p. 228).

[8] As one can see [for the validity of the principle of virtual work] any finite velocity can be taken... When one wants to measure these velocities for the spaces that the bodies cover in reality because these [velocities] vary instant to instant due to the constraints, one must then consider infinitesimally small spaces; if not they would not measure the given velocities anymore; and in this way one falls in the principle of virtual velocities where the principle loses a part of its clearness (Poinsot, Théorie générale de l'équilibre et du mouvement des systèmes, pp. 238–239).

[9] In this way the principle does not present anymore any trace of these ideas of infinitesimal displacements and perturbations of equilibrium, which seem strangers to the problem and which leave something obscure in the spirit (Poinsot, Théorie générale de l'équilibre et du mouvement des systèmes, p. 239).

comme on le suppose, il faudra qu'elles satisfassent aux équations,

$$f'(x)\frac{\mathrm{d}x}{\mathrm{d}t}+f'(y)\frac{\mathrm{d}y}{\mathrm{d}t}+f'(z)\frac{\mathrm{d}z}{\mathrm{d}t}+f'(x')\frac{\mathrm{d}x'}{\mathrm{d}t}+f'(y')\frac{\mathrm{d}y'}{\mathrm{d}t}+\&c.+\&c.=0.$$
$$\phi'(x)\frac{\mathrm{d}x}{\mathrm{d}t}+\phi(y)\frac{\mathrm{d}y}{\mathrm{d}t}+\phi'(z)\frac{\mathrm{d}z}{\mathrm{d}t}+\phi'(x')\frac{\mathrm{d}x'}{\mathrm{d}t}+\phi'(y')\frac{\mathrm{d}y'}{\mathrm{d}t}+\&c.+\&c.=0. \qquad \text{(B)}$$
$$\&c.$$

tirées des précédentes (A); il suffira qu'elles y satisfissent pour que les conditions de la liaison soient observées. Ou bien si l'on multiplie ces équations par des coefficiens quelconques indéterminés λ, μ, &c., et qu'on les ajoute, il suffira qu'elles satisfassent à la seule équation suivante indépendamment de λ, μ, &c.

$$\begin{aligned}&[\lambda f'(x)+\mu\phi'(x)+\&c.]\frac{\mathrm{d}x}{\mathrm{d}t}+[\lambda f'(y)+\mu\phi'(y)+\&c.]\frac{\mathrm{d}y}{\mathrm{d}t}+\\&+[\lambda f'(z)+\mu\phi'(z)+\&c.]\frac{\mathrm{d}z}{\mathrm{d}t}+[\lambda f'(x')+\mu\phi'(x')+\&c.]\frac{\mathrm{d}x'}{\mathrm{d}t}+\\&+[\lambda f'(y')+\mu\phi'(y')+\&c.]\frac{\mathrm{d}y'}{\mathrm{d}t}+\&c.+\&c.=0. \qquad \text{(C)}\end{aligned}$$

Or les fonctions qui multiplient les vitesses $(\mathrm{d}x/\mathrm{d}t)$, $(\mathrm{d}y/\mathrm{d}t)$, $(\mathrm{d}z/\mathrm{d}t)$, $(\mathrm{d}x'/\mathrm{d}t)$, &c., ne sont autre chose (après ce qui a été démontré) que les expressions générales des forces capables d'être en équilibre sur le système. Supposant donc des forces X, Y, Z; X', Y', Z', &c., feraient actuellement équilibre, on aurait:

$$X\frac{\mathrm{d}x}{\mathrm{d}t}+Y\frac{\mathrm{d}y}{\mathrm{d}t}+Z\frac{\mathrm{d}z}{\mathrm{d}t}+X'\frac{\mathrm{d}x'}{\mathrm{d}t}+Y'\frac{\mathrm{d}y'}{\mathrm{d}t}+Z'\frac{\mathrm{d}z'}{\mathrm{d}t}+\&c.+\&c.=0. \qquad \text{(D)}^{10}$$

[10] May the system be defined by the following equations between the coordinates of the bodies:

$$f(x,y,x;x',y',z',\&c.)=0. \qquad \phi(x,y,x;x',y',z',\&c.)=0. \qquad \&c. \qquad \text{(A)}$$

Supposing that any velocity is given to all these bodies that there can really be without violating the conditions of the constraints; the coordinates x, y, z, x', y', z', etc. will vary with time t, which we consider functions; and so that the given velocity $(\mathrm{d}x/\mathrm{d}t)$, $(\mathrm{d}y/\mathrm{d}t)$, $(\mathrm{d}z/\mathrm{d}t)$, $(\mathrm{d}x'/\mathrm{d}t)$, etc. are accepted by the constraints, as it had been supposed, it will be necessary that they satisfy the equations,

$$f'(x)\frac{\mathrm{d}x}{\mathrm{d}t}+f'(y)\frac{\mathrm{d}y}{\mathrm{d}t}+f'(z)\frac{\mathrm{d}z}{\mathrm{d}t}+f'(x')\frac{\mathrm{d}x'}{\mathrm{d}t}+f'(y')\frac{\mathrm{d}y'}{\mathrm{d}t}+\&c.+\&c.=0.$$
$$\phi'(x)\frac{\mathrm{d}x}{\mathrm{d}t}+\phi'(y)\frac{\mathrm{d}y}{\mathrm{d}t}+\phi'(z)\frac{\mathrm{d}z}{\mathrm{d}t}+\phi'(x')\frac{\mathrm{d}x'}{\mathrm{d}t}+\phi'(y')\frac{\mathrm{d}y'}{\mathrm{d}t}+\&c.+\&c.=0. \qquad \text{(B)}$$
$$\&c.$$

obtained from the previous one (A) and if they satisfy them it will be enough so that the conditions of constraints will be satisfied, if these equations are multiplied for any undetermined coefficients λ, μ, etc. and if they are summed up it will be sufficient for them [the velocities] to satisfy the single following condition, independently from λ, μ, etc.

$$\begin{aligned}&[\lambda f'(x)+\mu\phi'(x)+\&c.]\frac{\mathrm{d}x}{\mathrm{d}t}+[\lambda f'(y)+\mu\phi'(y)+\&c.]\frac{\mathrm{d}y}{\mathrm{d}t}+\\&+[\lambda f'(z)+\mu\phi'(z)+\&c.]\frac{\mathrm{d}z}{\mathrm{d}t}+[\lambda f'(x')+\mu\phi'(x')+\&c.]\frac{\mathrm{d}x'}{\mathrm{d}t}+\\&+[\lambda f'(y')+\mu\phi'(y')+\&c.]\frac{\mathrm{d}y'}{\mathrm{d}t}+\&c.+\&c.=0. \qquad \text{(C)}\end{aligned}$$

The functions that multiplied the velocities $(\mathrm{d}x/\mathrm{d}t)$, $(\mathrm{d}y/\mathrm{d}t)$, $(\mathrm{d}z/\mathrm{d}t)$, $(\mathrm{d}x'/\mathrm{d}t)$, (after what has been demonstrated) are nothing but the general expressions of the forces capable of being in equilibrium on the system and

Because Poinsot's virtual velocities, depending on the fact that they develop in virtual time, coincide with the virtual displacements introduced in the first part of this article, (D) represents the demonstration of the necessary part of the Theorem 3 that is, of VWP.

4. Conclusions

The problem of the logical status of VWL is complicated by the fact that till today no formulation exists which is universally accepted nor a defined framework of mechanics with which it is possible to confront ourselves. In this work I think I have shown under which point of view VWL may be considered a theorem of a given mechanics and under which conditions it must be considered an axiom (VWP). It appears clear that, as defined, the problem of demonstrability of VWP in a mechanics of reference is supplied by sense only if the constraints are characterised sufficiently in it. Only if this characterisation is complete, as when all constraints are assumed as smooth, to prove VWP is trivial. The characterisation of constraints cannot be based on *a priori* evidence but must have an empirical nature.

At the end of the XVIII century, after the publication of Lagrange's *Mécanique analytique*, VWL was at the centre of a lively debate; the demonstrations that had been attempted, even if they were interesting, did not appear to be definitive. Probably the most convincing one was Fourier's, which took VWL back to the law of lever; if this last one is accepted, then Fourier's reasoning can be considered conclusive. Due to lack of space, I was not able to present this debate in a detailed way and had to restrain myself from referring to the not so well-known contributions of the Italian school. But I dedicated some space to Poinsot's demonstration, because I believe that it influenced most the 'demonstrations' reported in the handbooks of statics, especially in the way it defines virtual displacements. Poinsot worked in the direction that I have called reductionist; first he defined completely his type of mechanics, and without introducing the questionable concept of constraint forces, he determined the conditions for the equilibrium of a system of points subject to internal and external constraints. After introducing in an appropriate way the virtual velocities, he had no difficulty in demonstrating VWP and the complete equivalence of the conclusions that ensued from it with those ensuing from the mechanics of reference.

In this paper I have dealt only with the case in which virtual displacements develop in virtual time; no attention has been given to when VWL is defined with displacements that develop in real time, that is with the same time with which the forces applied to the system vary. In this case the problem of the role of VWL appears more complex.

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nothing else. Supposing that the forces X, Y, Z, X', Y', Z', etc. balance each other, it is to have:

$$X\frac{\mathrm{d}x}{\mathrm{d}t}+Y\frac{\mathrm{d}y}{\mathrm{d}t}+Z\frac{\mathrm{d}z}{\mathrm{d}t}+X'\frac{\mathrm{d}x'}{\mathrm{d}t}+Y'\frac{\mathrm{d}y'}{\mathrm{d}t}+Z'\frac{\mathrm{d}z'}{\mathrm{d}t}+\&c.+\&c.=0. \tag{D}$$

(Poinsot, Théorie générale de l'équilibre et du mouvement des systèmes, pp. 237–238).

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